# NONLOCALITY AND THE CENTRAL GEOMETRY OF DIMER ALGEBRAS

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ABSTRACT. Let $A$ be a dimer algebra and $Z$ its center. It is well known that if $A$ is cancellative, then $A$ and $Z$ are noetherian and $A$ is a finitely generated $Z$-module. Here we show the converse: if $A$ is non-cancellative (as almost all dimer algebras are), then $A$ and $Z$ are nonnoetherian and $A$ is an infinitely generated $Z$-module. Although $Z$ is nonnoetherian, we show that it has Krull dimension 3 and is generically noetherian. Furthermore, we show that the reduced center is the coordinate ring for a Gorenstein algebraic variety with the strange property that it contains precisely one 'smeared-out' point of positive geometric dimension. In our proofs we introduce formalized notions of Higgsing and the mesonic chiral ring from quiver gauge theory.

## CONTENTS



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# 1. Introduction

1.1. **Overview.** We begin by recalling the definition of a dimer algebra, which is a type of quiver with potential whose quiver is dual to a dimer model.

**Definition 1.1.**

- Let $Q$ be a finite quiver whose underlying graph $\overline{Q}$ embeds into a two-dimensional real torus $T^2$, such that each connected component of $T^2 \setminus \overline{Q}$ is simply connected and bounded by an oriented cycle of length at least 2, called a *unit cycle*.[1] The *dimer algebra* of $Q$ is the quiver algebra $A = kQ/I$ with relations

(1) $$I := \langle p - q \mid \exists \ a \in Q_1 \text{ such that } pa \text{ and } qa \text{ are unit cycles} \rangle \subset kQ,$$

  where $p$ and $q$ are paths.
- $A$ and $Q$ are called *cancellative* if for all paths $p, q, r \in A$,
$$p = q \quad \text{whenever} \quad pr = qr \neq 0 \quad \text{or} \quad rp = rq \neq 0.$$

Cancellative dimer algebras are now well understood: they are 3-Calabi-Yau algebras and noncommutative crepant resolutions of 3-dimensional toric Gorenstein singularities (e.g., [Bo, Theorem 10.2], [Br], [D, Theorem 4.3], [MR, Theorem 6.3]).[2,3] Non-cancellative dimer algebras, on the other hand, are much less understood. However, almost all dimer algebras are non-cancellative, and so it is of great interest to understand them.

Our main results apply to non-cancellative dimer algebras that admit cyclic contractions, which is a new notion we introduce. Before stating our results, we define this notion.

Let $A = kQ/I$ be a (possibly non-cancellative) dimer algebra. Fix a subset of arrows $Q_1^* \subset Q_1$, and consider the quiver $Q'$ obtained by contracting each arrow in $Q_1^*$ to a vertex. This contraction defines a $k$-linear map of path algebras
$$\psi : kQ \to kQ'.$$
We call $\psi$ a *contraction of dimer algebras* if $Q'$ is also a dimer quiver, and $\psi$ induces a $k$-linear map of dimer algebras
$$\psi : A = kQ/I \to A' = kQ'/I',$$

[1] More generally, dimer algebras may be defined where $\overline{Q}$ embeds into any compact surface; see for example [Bo, Theorems 11.2 and 11.3] and [BKM]. However, here we will only consider dimer algebras for which $\overline{Q}$ embeds into a torus.

[2] A ring $A$ which is a finitely-generated module over a central normal Gorenstein subdomain $R$ is *Calabi-Yau of dimension* $n$ if (i) $\operatorname{gl.dim} A = \dim R = n$; (ii) $A$ is a maximal Cohen-Macaulay module over $R$; and (iii) $\operatorname{Hom}_R(A, R) \cong A$, as $A$-bimodules [Bra, Introduction].

[3] Let $A$ be a dimer algebra such that each arrow is contained in a perfect matching. Then $A$ is cancellative if and only if $A$ satisfies a combinatorial 'consistency condition' [IU, Theorem 1.1], [Bo, Theorem 6.2]. Cancellative dimer algebras are thus also called 'consistent'.

that is, $\psi(I) \subseteq I'$. An example is given in Figure 1.

We now describe the structure we wish to be preserved under a contraction.

**Definition 1.2.** Let $A = kQ/I$ be a dimer algebra.

- A *perfect matching* $D \subset Q_1$ is a set of arrows such that each unit cycle contains precisely one arrow in $D$.
- A *simple matching* $D \subset Q_1$ is a perfect matching such that $Q \setminus D$ supports a simple $A$-module of dimension $1^{Q_0}$ (that is, $Q \setminus D$ contains a cycle that passes through each vertex of $Q$). Denote by $\mathcal{S}$ the set of simple matchings of $A$.

Let $\psi : A \to A'$ be a contraction to a cancellative dimer algebra $A'$. Denote by

$$B := k\left[x_D \mid D \in \mathcal{S}'\right]$$

the polynomial ring generated by the simple matchings of $A'$. Denote by $E_{ij}$ a square matrix with a 1 in the $ij$-th slot and zeros elsewhere. Since $A'$ is cancellative, there is an algebra monomorphism

$$\tau : A' \hookrightarrow M_{|Q_0'|}(B) \tag{2}$$

defined on $i \in Q_0'$ and $a \in Q_1'$ by

$$\tau(e_i) := E_{ii}, \quad \tau(a) := E_{\mathrm{h}(a),\mathrm{t}(a)} \prod_{a \in D \in \mathcal{S}'} x_D,$$

and extended multiplicatively and $k$-linearly to $A'$ (Theorem 3.5). For $p \in e_j A' e_i$, denote by $\bar{\tau}(p) \in B$ the single nonzero matrix entry of $\tau(p)$, that is, $\tau(p) = \bar{\tau}(p) E_{ji}$. If

$$S := k\left[\cup_{i \in Q_0} \bar{\tau}\psi\left(e_i A e_i\right)\right] = k\left[\cup_{i \in Q_0'} \bar{\tau}\left(e_i A' e_i\right)\right],$$

then we say $\psi$ is *cyclic*, and call $S$ the *cycle algebra* of $A$.

Cyclic contractions are universal localizations that preserve the cycle algebra. Furthermore, they formalize the notion of Higgsing in abelian quiver gauge theories, where (morally) the mesonic chiral ring is preserved (Appendix A). The cycle algebra is isomorphic to the center $Z'$ of $A'$, and contains the reduced center

$$\hat{Z} := Z / \operatorname{nil} Z$$

of $A$ as a subalgebra (Theorems 3.5 and 4.27). Moreover, the cycle algebra is uniquely determined by $A$: it is independent of the choice of $\psi$. Indeed, $S$ is isomorphic to the GL-invariant functions on the Zariski-closure of the variety of simple $A$-modules of dimension $1^{Q_0}$ [B3, Theorem 3.13].

Our first main theorem is the following.

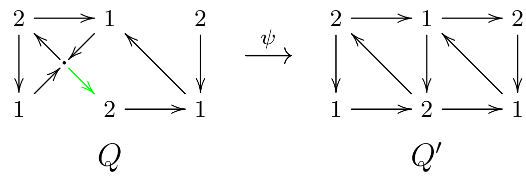


FIGURE 1. The non-cancellative dimer algebra $A = kQ/I$ cyclically contracts to the cancellative dimer algebra $A' = kQ'/I'$. Both quivers are drawn on a torus, and the contracted arrow is drawn in green. Here, the cycle algebra of $A$ is $S = k\left[x^2, y^2, xy, z\right] \subset B = k[x, y, z]$, and the homotopy center of $A$ is $R = k + (x^2, y^2, xy)S$.

**Theorem 1.3.** *Let $A$ be a dimer algebra with center $Z$, and suppose $A$ admits a cyclic contraction. Then the following are equivalent (Theorems 4.41 and 4.50):*

(1) *$A$ is cancellative.*
(2) *$A$ is noetherian.*
(3) *$Z$ is noetherian.*
(4) *$A$ is a finitely generated $Z$-module.*
(5) *The vertex corner rings $e_iAe_i$ are pairwise isomorphic algebras.*
(6) *Each vertex corner ring $e_iAe_i$ is isomorphic to $Z$.*
(7) *Each arrow annihilates a simple $A$-module of dimension $1^{Q_0}$.*
(8) *Each arrow is contained in a simple matching.*
(9) *If $\psi : A \to A'$ is a cyclic contraction, then $S = k\left[\cap_{i \in Q_0} \bar{\tau}\psi\left(e_iAe_i\right)\right]$.*
(10) *If $\psi : A \to A'$ is a cyclic contraction, then $\psi$ is trivial.*

We then use the notion of depiction and geometric dimension, introduced in [B2], to make sense of the central geometry of non-cancellative dimer algebras. A depiction of a nonnoetherian integral domain $R$ is a closely related noetherian overring $S$ that provides a way of visualizing the geometry of $R$ (Definition 1.9). The underlying idea is that nonnoetherian geometry is the geometry of *nonlocal* algebraic varieties and schemes. In this framework, (closed) points can be 'smeared-out', and thus have positive dimension. Such points are therefore inherently nonlocal. We use the term 'nonlocal' in the sense that two widely separated points may somehow be very near to each other.

Our second main theorem characterizes the central geometry of non-cancellative dimer algebras.

**Theorem 1.4.** *Let $A$ be a non-cancellative dimer algebra which admits a cyclic contraction. Then*

(1) *$Z$ and $\hat{Z} := Z/\operatorname{nil} Z$ each have Krull dimension 3 (Theorem 4.66).*
(2) *$\hat{Z}$ is a nonnoetherian integral domain depicted by the cycle algebra of $A$ (Corollary 4.28 and Theorem 4.68).*

(3) *The reduced induced scheme structure of* $\operatorname{Spec} Z$ *is birational to a noetherian affine scheme, and contains precisely one closed point of positive geometric dimension. Furthermore, the maximal ideal spectrum* $\operatorname{Max} \hat{Z}$ *may be viewed as a Gorenstein algebraic variety* $X$ *with one positive dimensional subvariety* $Y \subset X$ *identified as a single 'smeared-out' point (Theorem 4.68).*

In the course of proving Theorems 1.3 and 1.4, we also introduce the *homotopy dimer algebra* of $A$ (Definition 4.33),

$$\tilde{A} := A / \langle p - q \mid \exists \text{ path } r \text{ such that } pr = qr \neq 0 \text{ or } rp = rq \neq 0 \rangle ,$$

and the *homotopy center* of $A$ (Definitions 4.3 and 4.26),

$$R := k \left[ \cap_{i \in Q_0} \bar{\tau}\psi(e_i A e_i) \right].$$

We show that the center of the homotopy dimer algebra $\tilde{A}$ is isomorphic to the homotopy center $R$ of $A$ (Theorem 4.35). Furthermore, the reduced center $\hat{Z}$ of $A$ is a subalgebra of its homotopy center $R$ (Theorem 4.27). In particular, the kernel of $\psi$, restricted to $Z$, coincides with the nilradical of $Z$ (Theorem 4.24),

$$\ker \psi \cap Z = \operatorname{nil} Z.$$

In general, the containment $\hat{Z} \subseteq R$ may be proper (Theorem 4.31). Even so, $\hat{Z}$ and $R$ determine the same nonlocal variety (Theorem 4.68). Furthermore, their integral closures coincide (Theorem 4.75). Finally, we give necessary and sufficient conditions for $R$ to be normal; for instance, $R$ is normal if and only if there is an ideal $J$ of $S$ such that $R = k + J$ (Theorem 4.74).

*Conventions:* Throughout, $k$ is an uncountable algebraically closed field of characteristic zero. Let $R$ be an integral domain and a $k$-algebra. We will denote by $\dim R$ the Krull dimension of $R$; by $\operatorname{Frac} R$ the ring of fractions of $R$; by $\operatorname{Max} R$ the set of maximal ideals of $R$; by $\operatorname{Spec} R$ either the set of prime ideals of $R$ or the affine $k$-scheme with global sections $R$; by $R_{\mathfrak{p}}$ the localization of $R$ at $\mathfrak{p} \in \operatorname{Spec} R$; and by $\mathcal{Z}(\mathfrak{a})$ the closed set $\{\mathfrak{m} \in \operatorname{Max} R \mid \mathfrak{m} \supseteq \mathfrak{a}\}$ of $\operatorname{Max} R$ defined by the subset $\mathfrak{a} \subset R$.

We will denote by $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$ a quiver with vertex set $Q_0$, arrow set $Q_1$, and head and tail maps $\mathrm{h}, \mathrm{t} : Q_1 \to Q_0$. We will denote by $kQ$ the path algebra of $Q$, and by $e_i$ the idempotent corresponding to vertex $i \in Q_0$. Multiplication of paths is read right to left, following the composition of maps. We say a ring is noetherian if it is both left and right noetherian. By module we mean left module. By infinitely generated $R$-module, we mean an $R$-module that is not finitely generated. By nonconstant monomial, we mean a monomial that is not in $k$. Finally, we will denote by $E_{ij} \in M_d(k)$ the $d \times d$ matrix with a 1 in the $ij$-th slot and zeros elsewhere.

## 1.2. **Preliminaries.**

1.2.1. *Algebra homomorphisms from perfect matchings.* In this subsection, $A = kQ/I$ is a dimer algebra. Denote by $\mathcal{P}$ and $\mathcal{S}$ the sets of perfect and simple matchings of $A$ respectively. The following lemma is clear (see e.g., [MR, Section 4]).

**Lemma 1.5.** *Let $A = kQ/I$ be a dimer algebra. If $\sigma_i, \sigma_i'$ are two unit cycles at $i \in Q_0$, then $\sigma_i = \sigma_i'$. Furthermore, the element $\sum_{i \in Q_0} \sigma_i$ is in the center of $A$.*

We will denote by $\sigma_i \in A$ the unique unit cycle at vertex $i$.

**Lemma 1.6.** *Consider the maps*

$$\tau : A \to M_{|Q_0|}\left(k\left[x_D \mid D \in \mathcal{S}\right]\right) \quad \text{and} \quad \eta : A \to M_{|Q_0|}\left(k\left[x_D \mid D \in \mathcal{P}\right]\right) \tag{3}$$

*defined on $i \in Q_0$ and $a \in Q_1$ by*

$$\begin{aligned} \tau(e_i) &:= E_{ii}, \qquad & \tau(a) &:= E_{\mathrm{h}(a),\mathrm{t}(a)} \textstyle\prod_{a \in D \in \mathcal{S}} x_D, \\ \eta(e_i) &:= E_{ii}, \qquad & \eta(a) &:= E_{\mathrm{h}(a),\mathrm{t}(a)} \textstyle\prod_{a \in D \in \mathcal{P}} x_D, \end{aligned} \tag{4}$$

*and extended multiplicatively and $k$-linearly to $A$. Then $\tau$ and $\eta$ are algebra homomorphisms. Furthermore, each unit cycle $\sigma_i \in e_iAe_i$ satisfies*

$$\tau\left(\sigma_i\right) = E_{ii} \prod_{D \in \mathcal{S}} x_D \quad \text{and} \quad \eta\left(\sigma_i\right) = E_{ii} \prod_{D \in \mathcal{P}} x_D. \tag{5}$$

*Proof.* If $a \in Q_1$ and $pa$, $qa$ are unit cycles, then

$$\tau(p) = E_{\mathrm{t}(a),\mathrm{h}(a)} \prod_{a \notin D \in \mathcal{S}} x_D = \tau(q).$$

$\tau$ is therefore well-defined by the relations $I$ given in (1), and thus clearly an algebra homomorphism. $\eta$ is similarly an algebra homomorphism.

By definition, each perfect matching contains precisely one arrow in each unit cycle. Therefore (5) holds. □

**Notation 1.7.** For each $i, j \in Q_0$, denote by

$$\bar{\tau} : e_jAe_i \to B := k\left[x_D \mid D \in \mathcal{S}\right] \quad \text{and} \quad \bar{\eta} : e_jAe_i \to k\left[x_D \mid D \in \mathcal{P}\right]$$

the respective $k$-linear maps defined on $p \in e_jAe_i$ by

$$\tau(p) = \bar{\tau}(p)E_{ji} \quad \text{and} \quad \eta(p) = \bar{\eta}(p)E_{ji}.$$

That is, $\bar{\tau}(p)$ and $\bar{\eta}(p)$ are the single nonzero matrix entries of $\tau(p)$ and $\eta(p)$ respectively. In Sections 2 and 3, we will set

$$\bar{p} := \bar{\tau}(p) \quad \text{and} \quad \sigma := \prod_{D \in \mathcal{S}} x_D \quad \text{(or occasionally, } \sigma := \prod_{D \in \mathcal{P}} x_D\text{)}. \tag{6}$$

In Section 4, we will consider a cyclic contraction $\psi : A \to A'$. Denote by $\mathcal{S}'$ the set of simple matchings of $A'$. For $p \in e_j A e_i$ and $q \in e_\ell A' e_k$, we will set

$$\bar{p} := \bar{\tau}\psi(p), \quad \bar{q} := \bar{\tau}(q), \quad \text{and} \quad \sigma := \prod_{D \in \mathcal{S}'} x_D. \tag{7}$$

1.2.2. *Impressions.* The following definition, introduced in [B], captures a useful matrix ring embedding.

**Definition 1.8.** [B, Definition 2.1] An *impression* $(\tau, B)$ of a finitely generated noncommutative algebra $A$ is a finitely generated commutative algebra $B$ and an algebra monomorphism $\tau : A \hookrightarrow M_d(B)$ such that

- for generic $\mathfrak{b} \in \operatorname{Max} B$, the composition
$$A \xrightarrow{\tau} M_d(B) \xrightarrow{\epsilon_{\mathfrak{b}}} M_d(B/\mathfrak{b}) \cong M_d(k) \tag{8}$$
is surjective; and
- the morphism $\operatorname{Max} B \to \operatorname{Max} \tau(Z)$, $\mathfrak{b} \mapsto \mathfrak{b} \cap \tau(Z)$, is surjective.

An impression determines the center of $A$ explicitly [B, Lemma 2.1],

$$Z \cong \{ f \in B \mid f 1_d \in \operatorname{im} \tau \} \subseteq B.$$

Furthermore, if $A$ is a finitely generated module over its center, then its impression classifies all simple $A$-module isoclasses of maximal $k$-dimension [B, Proposition 2.5]. Specifically, for each such module $V$, there is some $\mathfrak{b} \in \operatorname{Max} B$ such that

$$V \cong (B/\mathfrak{b})^d,$$

where $av := \epsilon_{\mathfrak{b}}(\tau(a))v$ for each $a \in A$, $v \in V$. If $A$ is nonnoetherian, then its impression may characterize the central geometry of $A$ using depictions [B2, Section 3].

Now let $A$ be a dimer algebra. We will show that $(\tau, k[x_D \mid D \in \mathcal{S}])$ in (3) is an impression of $A$ if $A$ is cancellative (Theorem 3.5). Furthermore, we will show that $A$ is cancellative if and only if $A$ admits an impression $(\tau, B)$, where $B$ is an integral domain and $\tau(e_i) = E_{ii}$ for each $i \in Q_0$ (Corollary 3.7). However, if $A$ is non-cancellative, then its 'homotopy algebra' admits such an impression (Definition 4.33 and Theorem 4.35). We will then use this impression to understand the nonnoetherian central geometry of a large class of non-cancellative dimer algebras in Section 4.5.

1.2.3. *Depictions.* The following definition, introduced in [B2], aims to capture the geometry of nonnoetherian algebras with finite Krull dimension.

**Definition 1.9.** [B2, Definition 2.11.] Let $S$ be an integral domain and a noetherian $k$-algebra. Let $R$ be a nonnoetherian subalgebra of $S$ which contains $S$ in its fraction field, and suppose there is a point $\mathfrak{m} \in \operatorname{Max} R$ such that $R_{\mathfrak{m}}$ is noetherian.

(1) We say $R$ is *depicted* by $S$ if
   (a) the morphism $\iota_{R,S} : \operatorname{Spec} S \to \operatorname{Spec} R$, $\mathfrak{q} \mapsto \mathfrak{q} \cap R$, is surjective, and

(b) for each $\mathfrak{n} \in \operatorname{Max} S$, if $R_{\mathfrak{n}\cap R}$ is noetherian, then $R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}$.

(2) The *geometric codimension* or *geometric height* of $\mathfrak{p} \in \operatorname{Spec} R$ is the infimum

$$\operatorname{ght}(\mathfrak{p}) := \inf \left\{ \operatorname{ht}(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{R,S}^{-1}(\mathfrak{p}), \ S \text{ a depiction of } R \right\}.$$

The *geometric dimension* of $\mathfrak{p}$ is the difference

$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght}(\mathfrak{p}).$$

We will consider the following subsets of the variety $\operatorname{Max} S$,

$$\begin{aligned} U^*_{R,S} &:= \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} \text{ is noetherian}\} \\ U_{R,S} &:= \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\}. \end{aligned} \tag{9}$$

The locus $U_{R,S}$ specifies where $\operatorname{Max} R$ 'looks like' the variety $\operatorname{Max} S$. Furthermore, condition (1.b) in Definition 1.9 is equivalent to

$$U^*_{R,S} = U_{R,S}.$$

**Example 1.10.** Let $S = k[x,y]$, and consider its nonnoetherian subalgebra

$$R = k[x, xy, xy^2, \ldots] = k + xS.$$

$R$ is then depicted by $S$, and the closed point $xS \in \operatorname{Max} R$ has geometric dimension 1 [B2, Proposition 2.8 and Example 2.20]. Furthermore, $U_{R,S}$ is the complement of the line

$$\mathcal{Z}(x) = \{x = 0\} \subset \operatorname{Max} S.$$

In particular, $\operatorname{Max} R$ may be viewed as 2-dimensional affine space $\mathbb{A}^2_k = \operatorname{Max} S$ with the line $\mathcal{Z}(x)$ identified as a single 'smeared-out' point. From this perspective, $xS$ is a positive dimensional, hence nonlocal, point of $\operatorname{Max} R$.

In Section 4.5, we will show that both the reduced and homotopy centers of $A$, namely $\hat{Z}$ and $R$, are depicted by its cycle algebra $S$. Furthermore, we will find that $\hat{Z}$ and $R$ determine the the same nonlocal variety.

## 2. Cycle structure

Unless stated otherwise, $A = kQ/I$ is a dimer algebra with at least one perfect matching (Definition 1.2).[4] Furthermore, unless stated otherwise, by path or cycle we mean path or cycle modulo $I$. Throughout, we use the notation (6).

**Notation 2.1.** Let $\pi : \mathbb{R}^2 \to T^2$ be a covering map such that for some $i \in Q_0$,

$$\pi\left(\mathbb{Z}^2\right) = i \in Q_0.$$

Denote by $Q^+ := \pi^{-1}(Q) \subset \mathbb{R}^2$ the covering quiver of $Q$. For each path $p$ in $Q$, denote by $p^+$ the unique path in $Q^+$ with tail in the unit square $[0,1) \times [0,1) \subset \mathbb{R}^2$ satisfying $\pi(p^+) = p$.

[4] An example of a dimer algebra with no perfect matchings is given in [B3, Example 3.15].

Furthermore, for paths $p$, $q$ satisfying

$$\mathrm{t}(p^+) = \mathrm{t}(q^+) \quad \text{and} \quad \mathrm{h}(p^+) = \mathrm{h}(q^+), \tag{10}$$

denote by $\mathcal{R}_{p,q}$ the compact region in $\mathbb{R}^2 \supset Q^+$ bounded by (representatives of) $p^+$ and $q^+$.[5] Finally, denote by $\mathcal{R}^\circ_{p,q}$ the interior of $\mathcal{R}_{p,q}$.

**Definition 2.2.**

- We say say two paths $p, q \in A$ are a *non-cancellative pair* if $p \neq q$, and there is a path $r \in A$ such that
$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0.$$
- We say a non-cancellative pair $p, q$ is *minimal* if for each non-cancellative pair $s, t$,
$$\mathcal{R}_{s,t} \subseteq \mathcal{R}_{p,q} \quad \text{implies} \quad \{s, t\} = \{p, q\}.$$

**Lemma 2.3.** *Let $p, q \in e_jAe_i$ be distinct paths such that*

$$\mathrm{t}(p^+) = \mathrm{t}(q^+) \quad \textit{and} \quad \mathrm{h}(p^+) = \mathrm{h}(q^+). \tag{11}$$

*Then*

*(1) $p\sigma_i^m = q\sigma_i^n$ for some $m, n \geq 0$.*
*(2) $\bar{\tau}(p) = \bar{\tau}(q)\sigma^m$ for some $m \in \mathbb{Z}$.*
*(3) $\bar{\eta}(p) = \bar{\eta}(q)$ if and only if $p, q$ is a non-cancellative pair.*

*Proof.* (1) Suppose $p, q \in e_jAe_i$ satisfy (11). We proceed by induction on the region $\mathcal{R}_{p,q} \subset \mathbb{R}^2$ bounded by $p^+$ and $q^+$, with respect to inclusion.

If there are unit cycles $sa$ and $ta$ with $a \in Q_1$, and $\mathcal{R}_{p,q} = \mathcal{R}_{s,t}$, then the claim is clear. So suppose the claim holds for all pairs of paths $s, t$ such that

$$\mathrm{t}(s^+) = \mathrm{t}(t^+), \quad \mathrm{h}(s^+) = \mathrm{h}(t^+), \quad \text{and} \quad \mathcal{R}_{s,t} \subset \mathcal{R}_{p,q}.$$

Factor $p$ and $q$ into subpaths that are maximal length subpaths of unit cycles,

$$p = p_m p_{m-1} \cdots p_1 \quad \text{and} \quad q = q_n q_{n-1} \cdots q_1.$$

Then for each $1 \leq i \leq m$ and $1 \leq j \leq n$, there are paths $p'_i$ and $q'_j$ such that

$$p'_i p_i \quad \text{and} \quad q'_j q_j$$

are unit cycles, and $p_i^+$ and $q_j^+$ lie in the region $\mathcal{R}_{p,q}$. See Figure 2. Note that if $p_j$ is itself a unit cycle, then $p'_j$ is the vertex $\mathrm{t}(p_j)$, and similarly for $q_j$.

Consider the paths

$$p' := p'_1 p'_2 \cdots p'_m \quad \text{and} \quad q' := q'_1 q'_2 \cdots q'_n.$$

Then by Lemma 1.5 there is some $c, d \geq 0$ such that

$$p'p = \sigma_i^c \quad \text{and} \quad q'q = \sigma_i^d. \tag{12}$$

[5]In Lemma 2.12.1, we will not require (10) to hold.

Now

$$\mathrm{t}(p'^{+}) = \mathrm{t}(q'^{+}), \quad \mathrm{h}(p'^{+}) = \mathrm{h}(q'^{+}), \quad \text{and} \quad \mathcal{R}_{p',q'} \subset \mathcal{R}_{p,q}.$$

Thus by induction there is some $c', d' \geq 0$ such that

$$p'\sigma_i^{c'} = q'\sigma_i^{d'}.$$

Therefore by Lemma 1.5,

$$p\sigma_i^{d+d'} = pq'q\sigma_i^{d'} = pp'q\sigma_i^{c'} = q\sigma_i^{c+c'},$$

proving our claim.

(2) By Claim (1) there is some $m, n \geq 0$ such that

$$p\sigma_i^m = q\sigma_i^n. \tag{13}$$

Thus by Lemma 1.6,

$$\bar{\tau}(p)\sigma^m = \bar{\tau}\left(p\sigma_i^m\right) = \bar{\tau}\left(q\sigma_i^n\right) = \bar{\tau}(q)\sigma^n \in B. \tag{14}$$

Claim (2) then follows since $B$ is an integral domain.

(3.i) First suppose $\bar{\eta}(p) = \bar{\eta}(q)$. Set $\sigma := \prod_{D \in \mathcal{P}} x_D$. Then (14) implies

$$\sigma^m = \sigma^n$$

since $B$ is an integral domain (with $\bar{\eta}$ in place of $\bar{\tau}$). Recall our standing assumption, $\mathcal{P} \neq \emptyset$. In particular, $\sigma \neq 1$. Therefore $m = n$. Consequently, the path

$$r = \sigma_j^m$$

satisfies $rp = rq \neq 0$ by (13) and Lemma 1.5.

(3.ii) Conversely, suppose $p, q$ is a non-cancellative pair. Then there is a path $r$ such that

$$rp = rq \neq 0 \quad \text{ or } \quad pr = qr \neq 0;$$

say $rp = rq$. Whence

$$\bar{\eta}(r)\bar{\eta}(p) = \bar{\eta}(rp) = \bar{\eta}(rq) = \bar{\eta}(r)\bar{\eta}(q),$$

by Lemma 1.6. Therefore $\bar{\eta}(p) = \bar{\eta}(q)$ since $B$ is an integral domain. □

**Lemma 2.4.**

*(1) Suppose paths $p, q$ are either equal modulo $I$, or form a non-cancellative pair. Then their lifts $p^+$ and $q^+$ bound a compact region $\mathcal{R}_{p,q}$ in $\mathbb{R}^2$.*

*(2) Suppose paths $p, q$ are equal modulo $I$. If $i^+$ is a vertex in $\mathcal{R}_{p,q}$, then there is a path $r^+$ from $\mathrm{t}(p^+)$ to $\mathrm{h}(p^+)$ that is contained in $\mathcal{R}_{p,q}$, passes through $i^+$, and satisfies*

$$p = r = q \quad \textit{(modulo } I\textit{)}. \tag{15}$$

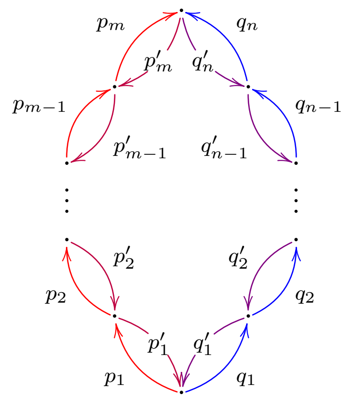


FIGURE 2. Setup for Lemma 2.3.1. The paths $p = p_m \cdots p_1$, $q = q_n \cdots q_1$, $p' = p'_1 \cdots p'_m$, and $q' = q'_1 \cdots q'_n$ are drawn in red, blue, purple, and violet respectively. Each product $p'_i p_i$ and $q'_j q_j$ is a unit cycle. Note that the region $\mathcal{R}_{p',q'}$ is properly contained in the region $\mathcal{R}_{p,q}$.

*Proof.* (1.i) First suppose $p, q$ are equal modulo $I$. The relations generated by the ideal $I$ in (1) lift to homotopy relations on the paths of $Q^+$. Thus $\mathrm{t}(p^+) = \mathrm{t}(q^+)$ and $\mathrm{h}(p^+) = \mathrm{h}(q^+)$. Therefore $p^+$ and $q^+$ bound a compact region in $\mathbb{R}^2$.

(1.ii) Now suppose $p, q$ is a non-cancellative pair. Then there is a path $r$ such that $rp = rq \neq 0$, say. In particular, $\mathrm{t}((rp)^+) = \mathrm{t}((rq)^+)$ and $\mathrm{h}((rp)^+) = \mathrm{h}((rq)^+)$ by Claim (1.i.). Therefore $\mathrm{t}(p^+) = \mathrm{t}(q^+)$ and $\mathrm{h}(p^+) = \mathrm{h}(q^+)$ as well.

(2) Recall that $I$ is generated by relations of the form $s - t$, where there is an arrow $a$ such that $sa$ and $ta$ are unit cycles. The claim then follows since each vertex subpath of the unit cycle $sa$ (resp. $ta$) is a vertex subpath of $s$ (resp. $t$). $\square$

**Definition 2.5.** A unit cycle $\sigma_i \in A$ of length 2 is a *2-cycle*. A 2-cycle is *removable* if the two arrows it is composed of are redundant generators for $A$, and otherwise the 2-cycle is *permanent*.

**Lemma 2.6.** *There are precisely two types of permanent 2-cycles. These are given in Figures 3.ii and 3.iii.*

*Proof.* Let $ab$ be a permanent 2-cycle, with $a, b \in Q_1$. Let

$$\sigma_{\mathrm{t}(a)} = sa \quad \text{and} \quad \sigma'_{\mathrm{t}(b)} = tb$$

be the complementary unit cycles to $ab$ containing $a$ and $b$ respectively. Since $ab$ is permanent, $b$ is a subpath of $s$, or $a$ is a subpath of $t$.

Suppose $b$ is a subpath of $s$. If $b$ is a leftmost subpath of $s$, that is, $s = bp$ for some path $p$, then we have the setup given in Figure 3.ii. Otherwise there is a non-vertex path $q$ such that $s = qbp$, in which case we have the setup given in Figure 3.iii. $\square$

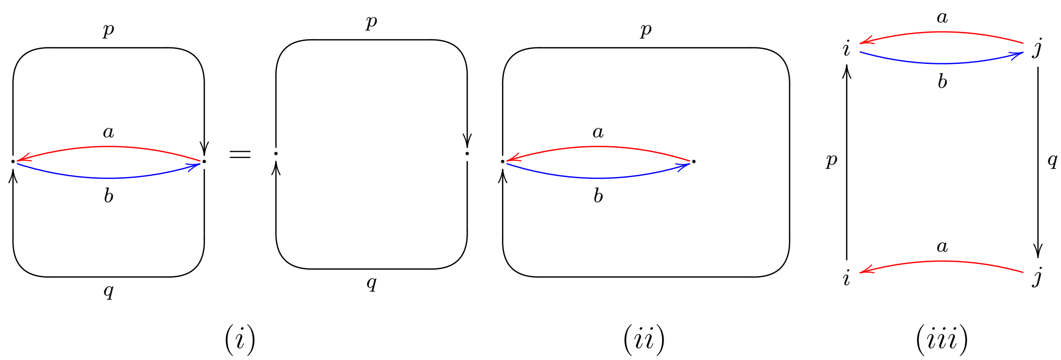


FIGURE 3. Cases for Lemma 2.6. In each case, $a$ and $b$ are arrows, and $p$ and $q$ are non-vertex paths. In case (i) $ap$, $bq$, and $ab$ are unit cycles; in case (ii) $abp$ and $ab$ are unit cycles; in case (iii) $qbpa$ and $ab$ are unit cycles, and $p$ and $q$ are cycles. In case (i) $ab$ is a removable 2-cycle, and in cases (ii) and (iii) $ab$ is a permanent 2-cycle.

**Notation 2.7.** By a *cyclic subpath* of a path $p$, we mean a proper subpath of $p$ that is a non-vertex cycle. Consider the following sets of cycles in $A$:

- Let $\mathcal{C}$ be the set of cycles in $A$ (i.e., cycles in $Q$ modulo $I$).
- For $u \in \mathbb{Z}^2$, let $\mathcal{C}^u$ be the set of cycles $p \in \mathcal{C}$ such that
$$\mathrm{h}(p^+) = \mathrm{t}(p^+) + u \in Q_0^+.$$
- For $i \in Q_0$, let $\mathcal{C}_i$ be the set of cycles in the vertex corner ring $e_iAe_i$.
- Let $\hat{\mathcal{C}}$ be the set of cycles $p \in \mathcal{C}$ such that $(p^2)^+$ does not have a cyclic subpath; or equivalently, the lift of each cyclic permutation of $p$ does not have a cyclic subpath.

We denote the intersection $\hat{\mathcal{C}} \cap \mathcal{C}^u \cap \mathcal{C}_i$, for example, by $\hat{\mathcal{C}}_i^u$. Note that $\mathcal{C}^0$ is the set of cycles whose lifts are cycles in $Q^+$. In particular, $\hat{\mathcal{C}}^0 = \emptyset$. Furthermore, the lift of any cycle $p$ in $\hat{\mathcal{C}}$ has no cyclic subpaths, although $p$ itself may have cyclic subpaths.

**Lemma 2.8.** *Suppose $A$ does not contain a non-cancellative pair where one of the paths is a vertex. Let $p$ be a non-vertex cycle.*

*(1) If $p \in \mathcal{C}^0$, then $\bar{p} = \sigma^m$ for some $m \geq 1$.*
*(2) If $p \in \mathcal{C}^0$ and $A$ is cancellative, then $p = \sigma_i^m$ for some $m \geq 1$.*
*(3) If $p \in \mathcal{C} \setminus \hat{\mathcal{C}}$, then $\sigma \mid \bar{p}$.*
*(4) If $p$ is a path for which $\sigma \nmid \bar{p}$, then $p$ is a subpath of a cycle in $\hat{\mathcal{C}}$.*

*Proof.* (1) Suppose $p \in \mathcal{C}^0$, that is, $p^+$ is a cycle in $Q^+$. Set $i := \mathrm{t}(p)$. By Lemma 2.3.1, there is some $m, n \geq 0$ such that

$$p\sigma_i^m = \sigma_i^n. \tag{16}$$

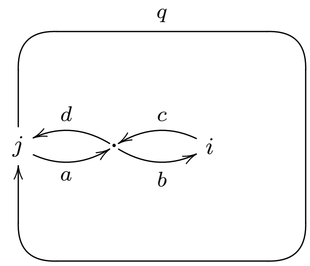


FIGURE 4. Setup for Remark 2.9. Let $m \geq 1$. Then the path $p := baq^{m+1}dc$ satisfies $p\sigma_i^m = \sigma_i$.

If $m \geq n$, then the paths $p\sigma_i^{m-n}$ and $e_{\mathrm{t}(p)}$ form a non-cancellative pair. But this is contrary to assumption. Thus $n - m \geq 1$.

Furthermore, $\bar{\tau}$ is an algebra homomorphism on $e_iAe_i$ by Lemma 1.6. In particular, (16) implies

$$\bar{p}\sigma^m = \sigma^n.$$

Therefore $\bar{p} = \sigma^{n-m}$ since $B$ is an integral domain.

(2) If $A$ is cancellative, then (16) implies $p = \sigma_i^{n-m}$.

(3) Suppose $p \in \mathcal{C} \setminus \hat{\mathcal{C}}$. Then there is a cyclic permutation of $p^+$ which contains a cyclic subpath $q^+$. In particular, $\bar{q} \mid \bar{p}$. Furthermore, since $q \in \mathcal{C}^0$, Claim (1) implies $\bar{q} = \sigma^m$ for some $m \geq 1$. Therefore $\sigma \mid \bar{p}$.

(4) Let $p$ be a path for which $\sigma \nmid \bar{p}$. Then there is a simple matching $D \in \mathcal{S}$ such that $x_D \nmid \bar{p}$. In particular, $p$ is supported on $Q \setminus D$. Since $D$ is simple, $p$ is a subpath of a cycle $q$ supported on $Q \setminus D$. Whence $x_D \nmid \bar{q}$. Thus $\sigma \nmid \bar{q}$. Therefore $q$ is in $\hat{\mathcal{C}}$ by the contrapositive of Claim (3). $\square$

**Remark 2.9.** In Lemma 2.8 we assumed that $A$ does not contain a non-cancellative pair where one of the paths is a vertex. Such non-cancellative pairs exist: if $A$ contains a permanent 2-cycle as in Figure 3.ii, then $p\sigma_{\mathrm{t}(p)} = \sigma_{\mathrm{t}(p)}$. In particular, $p, e_{\mathrm{t}(p)}$ is a non-cancellative pair.

Furthermore, it is possible for $m > n$ in (16). Consider a dimer algebra with the subquiver given in Figure 4. Here $a, b, c, d$ are arrows, $q$ is a non-vertex path, and $ad$, $bc$, $qdcba$ are unit cycles. Let $m \geq 1$, and set $p := baq^{m+1}dc$. Then

$$p\sigma_i^m = baq^{m+1}dc\sigma_i^m \overset{(\mathrm{I})}{=} ba(q\sigma_j)^m qdc = b(aqd)^{m+1}c = b(aqdc) = bc = \sigma_i,$$

where (I) holds by Lemma 1.5. Note, however, that such a dimer algebra does not admit a perfect matching.

By the definition of $\hat{\mathcal{C}}$, each cycle $p \in \mathcal{C}_i^u \setminus \hat{\mathcal{C}}$ has a representative $\tilde{p}$ that factors into subpaths $\tilde{p} = \tilde{p}_3\tilde{p}_2\tilde{p}_1$, where

$$p_1p_3 = \tilde{p}_1\tilde{p}_3 + I \in \mathcal{C}^0 \quad \text{and} \quad p_2 = \tilde{p}_2 + I \in \hat{\mathcal{C}}^u. \tag{17}$$

**Proposition 2.10.** *If $\hat{\mathcal{C}}_i^u = \emptyset$ for some $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$, then $A$ is non-cancellative.*

*Proof.* In the following, we will use the notation (17). Set $\sigma := \prod_{D \in \mathcal{P}} x_D$.

Suppose $\hat{\mathcal{C}}_i^u = \emptyset$. Let $p, q \in \mathcal{C}_i^u$ be cycles such that the region

$$\mathcal{R}_{\tilde{p}_3\tilde{p}_2^2\tilde{p}_1, \tilde{q}_3\tilde{q}_2^2\tilde{q}_1}$$

contains the vertex $i^+ + u \in Q_0^+$. Furthermore, suppose $p$ and $q$ admit representatives $\tilde{p}'$ and $\tilde{q}'$ (possibly distinct from $\tilde{p}$ and $\tilde{q}$) such that the region $\mathcal{R}_{\tilde{p}',\tilde{q}'}$ has minimal area among all such pairs of cycles $p, q$. See Figure 5.

By Lemmas 1.6 and 2.3.1, there is some $m \in \mathbb{Z}$ such that

$$\bar{\eta}(p_3p_2^2p_1) = \bar{\eta}(q_3q_2^2q_1)\sigma^m.$$

Suppose $m \geq 0$. Set

$$s := p_3p_2^2p_1 \quad \text{and} \quad t := q_3q_2^2q_1\sigma_i^m.$$

Then

$$\bar{\eta}(s) = \bar{\eta}(t). \tag{18}$$

Assume to the contrary that $A$ is cancellative. Then $s = t$ by Lemma 2.3.3. Furthermore, there is a path $r^+$ in $\mathcal{R}_{\tilde{s},\tilde{t}}$ which passes through the vertex

$$i^+ + u \in Q_0^+$$

and is homotopic to $s^+$ (by the relations $I$), by Lemma 2.4.2. In particular, $r$ factors into paths $r = r_2e_ir_1 = r_2r_1$, where

$$r_1, r_2 \in \mathcal{C}_i^u.$$

But $p$ and $q$ were chosen so that the area of $\mathcal{R}_{\tilde{p}',\tilde{q}'}$ is minimal. Thus there is some $\ell_1, \ell_2 \geq 0$ such that

$$\tilde{r}_1 = \tilde{p}'\sigma_i^{\ell_1} \quad \text{and} \quad \tilde{r}_2 = \tilde{p}'\sigma_i^{\ell_2} \quad (\text{modulo } I).$$

Set $\ell := \ell_1 + \ell_2$. Then

$$r = r_2r_1 = p^2\sigma_i^{\ell_1+\ell_2} = p^2\sigma_i^{\ell}. \tag{19}$$

Since $A$ is cancellative, the $\bar{\eta}$-image of any non-vertex cycle in $Q^+$ is a positive power of $\sigma$ by Lemma 2.8.1 (with $\bar{\eta}$ in place of $\bar{\tau}$). In particular, since $(p_1p_3)^+$ is a non-vertex cycle, there is an $n \geq 1$ such that

$$\bar{\eta}(p_1p_3) = \sigma^n. \tag{20}$$

Hence

$$\bar{\eta}(p)\bar{\eta}(p_2) \overset{\text{(I)}}{=} \bar{\eta}(s) = \bar{\eta}(r) \overset{\text{(II)}}{=} \bar{\eta}(p^2)\sigma^{\ell} \overset{\text{(III)}}{=} \bar{\eta}(p)\bar{\eta}(p_2)\bar{\eta}(p_1p_3)\sigma^{\ell} \overset{\text{(IV)}}{=} \bar{\eta}(p)\bar{\eta}(p_2)\sigma^{n+\ell}.$$

Indeed, (I) and (III) hold by Lemma 1.6, (II) holds by (19), and (IV) holds by (20). Furthermore, the image of $\bar{\eta}$ is in the integral domain $k[x_D \mid D \in \mathcal{P}]$. Thus

$$\sigma^{n+\ell} = 1.$$

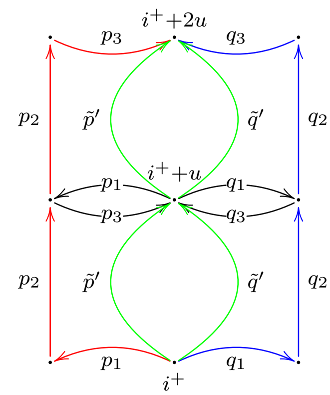


FIGURE 5. Setup for Proposition 2.10. The (lifts of the) cycles $s = p_3p_2^2p_1$ and $q_3q_2^2q_1$ are drawn in red and blue respectively. The representatives $\tilde{p}'$ and $\tilde{q}'$ of $p$ and $q$, such that the region $\mathcal{R}_{\tilde{p}',\tilde{q}'}$ has minimal area, are drawn in green.

But $n \geq 1$ and $\ell \geq 0$. Whence $\sigma = 1$. Therefore $Q$ has no perfect matchings, contrary to our standing assumption that $Q$ has at least one perfect matching. □

**Definition 2.11.** We call the subquiver given in Figure 6.i a *column*, and the subquiver given in Figure 6.ii a *pillar*. In the latter case, $\mathrm{h}(a_\ell)$ and $\mathrm{t}(a_1)$ are either vertex subpaths of $q_\ell$ and $p_1$ respectively, or $p_\ell$ and $q_1$ respectively.

**Lemma 2.12.** *Suppose paths $p^+$, $q^+$ have no cyclic subpaths, and bound a region $\mathcal{R}_{p,q}$ which contains no vertices in its interior.*

*(1) If $p$ and $q$ do not intersect, then $p^+$ and $q^+$ bound a column.*
*(2) Otherwise $p^+$ and $q^+$ bound a union of pillars. In particular, if*

$$\mathrm{t}(p^+) = \mathrm{t}(q^+) \quad \textit{and} \quad \mathrm{h}(p^+) = \mathrm{h}(q^+) \neq \mathrm{t}(p^+),$$

*then $p = q$ (modulo $I$).*

*Proof.* Suppose the hypotheses hold. Since $\mathcal{R}_{p,q}$ contains no vertices in its interior, each path that intersects its interior is an arrow. Thus $p^+$ and $q^+$ bound a union of subquivers given by the four cases in Figure 6.

In case (i),

$$p = p_\ell \cdots p_1 \quad \text{and} \quad q = q_\ell \cdots q_1.$$

In cases (ii) - (iv), the paths $p_\ell \cdots p_1$ and $q_\ell \cdots q_1$ are not-necessarily-proper subpaths of $p$ and $q$ respectively. In cases (ii) and (iv), $\mathrm{h}(a_\ell)$ and $\mathrm{t}(a_1)$ are either vertex subpaths of $q$ and $p$ respectively, or $p$ and $q$ respectively. In case (iii), $\mathrm{h}(a_\ell)$ and $\mathrm{t}(a_1)$ are either both vertex subpaths of $q$, or both vertex subpaths of $p$. In all cases,

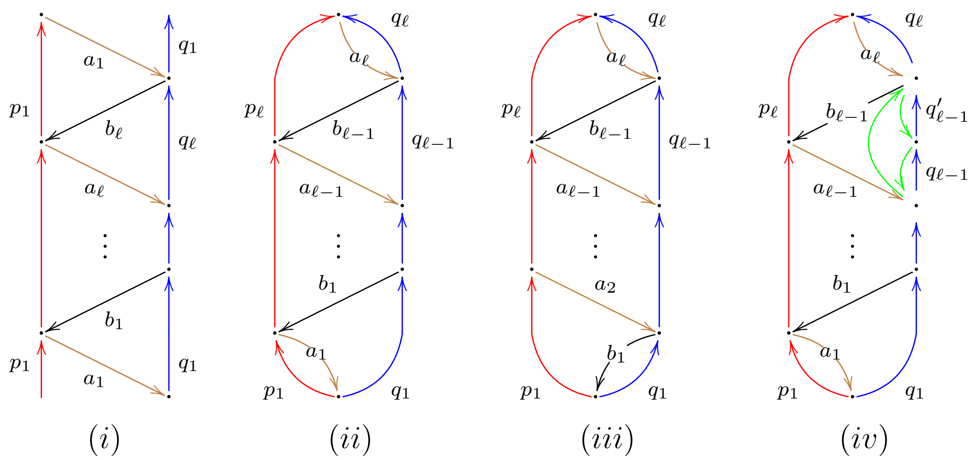


FIGURE 6. Cases for Lemmas 2.12 and 2.15. The subquiver in case (i) is a 'column', and the subquiver in case (ii) is a 'pillar'. The paths $a_1, \ldots, a_\ell, b_1, \ldots, b_\ell$ are arrows, and the paths $p_\ell \cdots p_1$ and $q_\ell \cdots q_1$, drawn in red and blue respectively, are not-necessarily-proper subpaths of $p$ and $q$. In case (iv), the green paths are also arrows. In cases (i), (ii), (iii), the cycles $q_j a_j b_j$ and $a_j p_j b_{j-1}$ are unit cycles. Note that $q^+$ has a cyclic subpath in cases (iii) and (iv), contrary to assumption. Furthermore, $pb_\ell = b_\ell q$ in case (i), and $p_\ell \cdots p_1 = q_\ell \cdots q_1$ in case (ii).

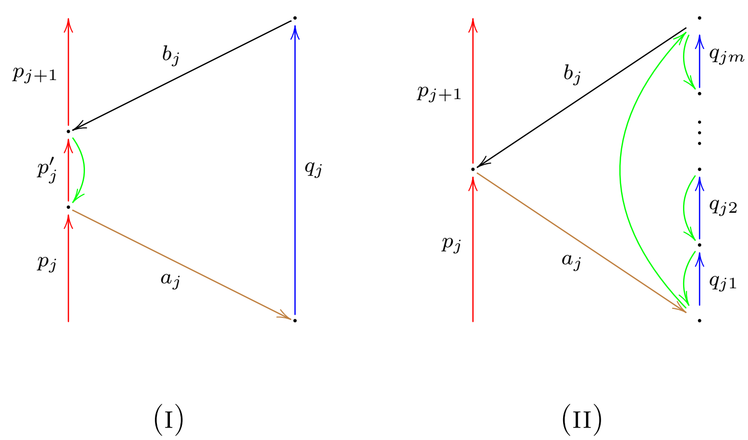


FIGURE 7. Generalizations of the setup given in Figure 6.iv. In both cases, the green paths are arrows. Case (I) shows that if $\mathrm{t}(a_j) \neq \mathrm{h}(b_j)$ (resp. $\mathrm{h}(a_j) \neq \mathrm{t}(b_{j-1})$), then $p^+$ (resp. $q^+$) has a cyclic subpath. In case (II), $q_j = q_{jm} \cdots q_{j2} q_{j1}$. This case shows that if $q_j a_j b_j$ (resp. $a_j p_j b_{j-1}$) is not a unit cycle, then $q^+$ (resp. $p^+$) has a cyclic subpath.

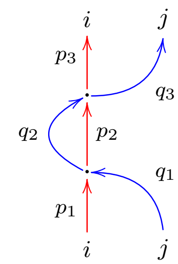


FIGURE 8. Setup for Lemma 2.14.

each cycle which bounds a region with no arrows in its interior is a unit cycle. In particular, in cases (i) - (iii), each path $a_j p_j b_{j-1}$, $b_j q_j a_j$ is a unit cycle.

Observe that in cases (iii) and (iv), $q^+$ has a cyclic subpath, contrary to assumption. Generalizations of case (iv) are considered in Figures 7.I and 7.II. Consequently, $p^+$ and $q^+$ bound either a column or a union of pillars. $\square$

**Notation 2.13.** For $u \in \mathbb{Z}^2$, denote by $\widehat{\mathcal{C}}^u \subseteq \hat{\mathcal{C}}^u$ a maximal subset of cycles in $\hat{\mathcal{C}}^u$ whose lifts do not intersect transversely in $\mathbb{R}^2$ (though they may share common subpaths).

In the following two lemmas, fix $u \in \mathbb{Z}^2 \setminus 0$ and a subset $\widehat{\mathcal{C}}^u \subseteq \hat{\mathcal{C}}^u$.

**Lemma 2.14.** *Suppose $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$. Then $\widehat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$.*

*Proof.* Suppose $p, q \in \hat{\mathcal{C}}^u$ intersect transversely at $k \in Q_0$. Then their lifts $p^+$ and $q^+$ intersect at least twice since $p$ and $q$ are both in $\mathcal{C}^u$. Thus $p$ and $q$ factor into paths $p = p_3 p_2 p_1$ and $q = q_3 q_2 q_1$, where

$$\mathrm{t}(p_2^+) = \mathrm{t}(q_2^+) \quad \text{and} \quad \mathrm{h}(p_2^+) = \mathrm{h}(q_2^+),$$

and $k^+$ is the tail or head of $p_2^+$. See Figure 8.

Since $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$, we may suppose that $\mathcal{R}_{p_2,q_2}$ contains no vertices in its interior. Since $p$ and $q$ are in $\hat{\mathcal{C}}$, $p_2^+$ and $q_2^+$ do not have cyclic subpaths. Thus $p_2 = q_2$ (modulo $I$) by Lemma 2.12.2. Therefore the paths

$$s := q_3 p_2 q_1 \quad \text{and} \quad t := p_3 q_2 p_1$$

equal $q$ and $p$ (modulo $I$) respectively. In particular, $s$ and $t$ are in $\hat{\mathcal{C}}^u$. The lemma then follows since $s^+$ and $t^+$ do not intersect transversely at the vertices $\mathrm{t}(p_2^+)$ or $\mathrm{h}(p_2^+)$. $\square$

**Lemma 2.15.** *Suppose $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$. Then there is a simple matching $D \in \mathcal{S}$ such that $Q \setminus D$ supports each cycle in $\widehat{\mathcal{C}}^u$.*

*Furthermore, if $A$ contains a column, then there are two simple matchings $D_1, D_2 \in \mathcal{S}$ such that $Q \setminus (D_1 \cup D_2)$ supports each cycle in $\widehat{\mathcal{C}}^u$.*

*Proof.* Suppose the hypotheses hold. By Lemma 2.14, $\widehat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$ since $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$. Thus we may consider cycles $p, q \in \widehat{\mathcal{C}}^u$ for which $\pi^{-1}(p)$ and $\pi^{-1}(q)$ bound a region $\mathcal{R}_{p,q}$ with no vertices in its interior.

Recall Figure 6. Since $\widehat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$, we may partition $Q$ into columns and pillars, by Lemma 2.12. Consider the subset of arrows $D_1$ (resp. $D_2$) consisting of all the $a_j$ arrows in each pillar of $Q$, and all the $a_j$ arrows (resp. $b_j$ arrows) in each column of $Q$. Note that $D_1$ consists of all the right-pointing arrows in each column, and $D_2$ consists of all the left-pointing arrows in each column. Furthermore, if $Q$ does not contain a column, then $D_1 = D_2$.

Observe that each unit cycle in each column and pillar contains precisely one arrow in $D_1$, and one arrow in $D_2$. (Note that this is not true for cases (iii) and (iv).) Furthermore, no such arrow occurs on the boundary of these regions, that is, as a subpath of $p$ or $q$. Therefore $D_1$ and $D_2$ are perfect matchings of $Q$.

Recall that a simple $A$-module of dimension $1^{Q_0}$ is characterized by the property that there is a non-annihilating path between any two vertices of $Q$. Clearly $Q \setminus D_1$ and $Q \setminus D_2$ each support a path that passes through each vertex of $Q$. Therefore $D_1$ and $D_2$ are simple matchings. □

**Lemma 2.16.** *If $A$ is cancellative, then $A$ has at least one simple matching.*

*Proof.* Follows from Proposition 2.10 and Lemma 2.15. □

Lemma 2.16 will be superseded by Theorems 2.24 and 4.41, where we will show that $A$ is cancellative if and only if each arrow in $Q$ is contained in a simple matching.

**Lemma 2.17.** *Let $u \in \mathbb{Z}^2 \setminus 0$, and suppose $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$. Then $A$ does not contain a non-cancellative pair where one of the paths is a vertex.*

*Proof.* Suppose $e_i, p$ is a non-cancellative pair. Then there is a path $r$ such that $r = pr \neq 0$. Whence

$$\mathrm{t}(r^+) = \mathrm{t}((pr)^+) \quad \text{and} \quad \mathrm{h}(r^+) = \mathrm{h}((pr)^+),$$

by Lemma 2.4.1. In particular,

$$\bar{r} = \sigma^m \bar{r}$$

for some $m \in \mathbb{Z}$, by Lemma 2.3.2. Thus $\sigma = 1$ since $B$ is an integral domain. Therefore $A$ has no simple matchings, $\mathcal{S} = \emptyset$. Consequently, for each $u \in \mathbb{Z}^2 \setminus 0$ there is a vertex $i \in Q_0$ such that $\hat{\mathcal{C}}^u_i = \emptyset$, by Lemma 2.15. □

**Lemma 2.18.** *Let $u \in \mathbb{Z}^2 \setminus 0$. If $p, q \in \mathcal{C}^u$, then there is some $n \in \mathbb{Z}$ such that $\bar{p} = \bar{q}\sigma^n$. In particular, if $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$, then $\bar{p} = \bar{q}$.*

*Proof.* Suppose the hypotheses hold. Since $Q$ is a dimer quiver, there is a path $r$ from $\mathrm{t}(p)$ to $\mathrm{t}(q)$. Thus there is some $m, n \geq 0$ such that

$$rp\sigma^m_{\mathrm{t}(p)} = qr\sigma^n_{\mathrm{t}(p)},$$

by Lemma 2.3.1. Furthermore, $\tau$ is an algebra homomorphism by Lemma 1.6. Thus

$$\bar{r}\bar{p}\sigma^m = \bar{\tau}\left(rp\sigma^n_{\mathrm{t}(p)}\right) = \bar{\tau}\left(qr\sigma^n_{\mathrm{t}(p)}\right) = \bar{q}\bar{r}\sigma^n.$$

Therefore $\bar{p} = \bar{q}\sigma^{n-m}$ since $B$ is an integral domain. $\square$

**Lemma 2.19.** *Suppose $A$ is cancellative.*

*(1) If $a \in Q_1$, $p \in \hat{\mathcal{C}}^u_{\mathrm{t}(a)}$, and $q \in \hat{\mathcal{C}}^u_{\mathrm{h}(a)}$, then $ap = qa$.*
*(2) Each vertex corner ring $e_iAe_i$ is commutative.*

*Proof.* (1) Suppose $a \in Q_1$, $p \in \hat{\mathcal{C}}^u_{\mathrm{t}(a)}$, and $q \in \hat{\mathcal{C}}^u_{\mathrm{h}(a)}$. Then

$$\mathrm{t}\left((ap)^+\right) = \mathrm{t}\left((qa)^+\right) \quad \text{and} \quad \mathrm{h}\left((ap)^+\right) = \mathrm{h}\left((qa)^+\right).$$

Let $r^+$ be a path in $Q^+$ from $\mathrm{h}\left((ap)^+\right)$ to $\mathrm{t}\left((ap)^+\right)$. Then by Lemma 2.8.2, there is some $m, n \geq 1$ such that

$$rap = \sigma^m_{\mathrm{t}(a)} \quad \text{and} \quad rqa = \sigma^n_{\mathrm{t}(a)}.$$

Assume to the contrary that $m < n$. Then $qa = ap\sigma^{n-m}_{\mathrm{t}(a)}$ since $A$ is cancellative. Let $b$ be a path such that $ab$ is a unit cycle. Then

$$q\sigma_{\mathrm{h}(a)} = qab = ap\sigma^{n-m}_{\mathrm{t}(a)}b = apb\sigma^{n-m}_{\mathrm{h}(a)}.$$

Thus, since $A$ is cancellative and $n - m \geq 1$,

$$q = apb\sigma^{n-m-1}_{\mathrm{h}(a)}.$$

Furthermore, $(ba)^+$ is cycle in $Q^+$ since $ba$ is a unit cycle. But this is a contradiction since $q$ is in $\hat{\mathcal{C}}^u$. Whence $m = n$. Therefore $qa = ap$.

(2) Since $I$ is generated by binomials, it suffices to show that if $p, q \in e_iAe_i$ are cycles, then $qp = pq$. Let $r^+$ be a path in $Q^+$ from $\mathrm{h}\left((qp)^+\right)$ to $\mathrm{t}\left((qp)^+\right)$. Set $r := \pi(r^+)$. Then $(rqp)^+$ and $(rpq)^+$ are cycles. In particular, there is some $m, n \geq 1$ such that

$$rqp = \sigma^m_i \quad \text{and} \quad rpq = \sigma^n_i,$$

by Lemma 2.8.2. Thus, since $\tau$ is an algebra homomorphism,

$$\sigma^m = \overline{rqp} = \bar{r}\bar{q}\bar{p} = \bar{r}\bar{p}\bar{q} = \overline{rpq} = \sigma^n.$$

Furthermore, $\sigma \neq 1$ by Lemma 2.16. Whence $m = n$ since $B$ is an integral domain. Thus $rqp = \sigma^m_i = rpq$. Therefore $qp = pq$ since $A$ is cancellative. $\square$

**Proposition 2.20.** *Let $u \in \mathbb{Z}^2 \setminus 0$. Suppose (i) $A$ is cancellative, or (ii) $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $i \in Q_0$.*

*(1) If $p \in \hat{\mathcal{C}}^u$, then $\sigma \nmid \bar{p}$.*
*(2) If $p, q \in \hat{\mathcal{C}}^u$, then $\bar{p} = \bar{q}$.*
*(3) If a cycle $p$ is formed from subpaths of cycles in $\hat{\mathcal{C}}^u$, then $p \in \hat{\mathcal{C}}$.*
*(4) If $p, q \in \hat{\mathcal{C}}^u_i$, then $p = q$.*

Note that in Claim (2) the cycles $p$ and $q$ are based at the same vertex $i$, whereas in Claim (3) $p$ and $q$ may be based at different vertices.

*Proof.* If $A$ is cancellative, then $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $i \in Q_0$, by Proposition 2.10. Therefore assumption (i) implies assumption (ii), and so it suffices to suppose (ii) holds.

(1) By Lemma 2.14, it suffices to consider $p \in \hat{\mathcal{C}}^u \setminus \widehat{\mathcal{C}}^u$. Then there are cycles $s, t \in \widehat{\mathcal{C}}^u$ such that

$$s = q_3 p_2 q_1, \quad t = p_3 q_2 p_1, \quad p = p_3 p_2 p_1,$$

as in Figure 8. In particular, $\sigma \nmid \bar{s}$ and $\sigma \nmid \bar{t}$ by Lemma 2.15. Thus

$$\bar{s} = \bar{t} \tag{21}$$

by Lemma 2.18. Set $q := q_3 q_2 q_1$.

Assume to the contrary that $\sigma \mid \bar{p}$. Then

$$\sigma \mid \bar{p}\bar{q} = \bar{p}_3\bar{p}_2\bar{p}_1\bar{q}_3\bar{q}_2\bar{q}_1 = \bar{s}\bar{t} \overset{(\mathrm{I})}{=} \bar{s}^2,$$

where (I) holds by (21). Therefore $\sigma \mid \bar{s}$ since $\sigma = \prod_{D \in \mathcal{S}} x_D$. But this is a contradiction to Lemma 2.15 since $s \in \widehat{\mathcal{C}}^u$.

(2) Follows from Claim (1) and Lemma 2.18.

A direct proof assuming $A$ is cancellative: Suppose $p, q \in \hat{\mathcal{C}}^u$. Let $r$ be a path from $\mathrm{t}(p)$ to $\mathrm{t}(q)$. Then $rp = qr$ by Lemma 2.19.1. Thus

$$\bar{r}\bar{p} = \overline{rp} = \overline{qr} = \bar{q}\bar{r} = \bar{r}\bar{q}.$$

Therefore $\bar{p} = \bar{q}$ since $B$ is an integral domain.

(3) Let $p = p_\ell \cdots p_1 \in \mathcal{C}^u$ be a cycle formed from subpaths $p_j$ of cycles $q_j$ in $\hat{\mathcal{C}}^u$. Then

$$g := \bar{q}_1 = \cdots = \bar{q}_\ell$$

by Claim (2). Furthermore, $\sigma \nmid g$ by Claim (1). In particular, there is a simple matching $D \in \mathcal{S}$ for which $x_D \nmid g$. Whence $x_D \nmid \bar{p}_j$ for each $1 \leq j \leq \ell$. Thus $x_D \nmid \bar{p}$. Therefore $\sigma \nmid \bar{p}$. Consequently, $p \in \hat{\mathcal{C}}$ by the contrapositive of Lemma 2.8.3, with Lemma 2.17.

(4) Follows from Claim (3) and Figure 6.ii.

A direct proof assuming $A$ is cancellative: Suppose $p, q \in \hat{\mathcal{C}}_i^u$. Let $r^+$ be a path in $Q^+$ from $\mathrm{h}(p^+)$ to $\mathrm{t}(p^+)$. Then there is some $m, n \geq 1$ such that

$$rp = \sigma_i^m \quad \text{and} \quad rq = \sigma_i^n,$$

by Lemma 2.8.2. Suppose $m \leq n$. Then $rp\sigma_i^{n-m} = \sigma_i^n = rq$. Thus $p\sigma_i^{n-m} = q$ since $A$ is cancellative. But then Claim (1) implies $m = n$ since by assumption $p$ and $q$ are in $\hat{\mathcal{C}}^u$. Therefore $p = q$. □

**Lemma 2.21.** *Suppose $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$. There is an $n \gg 1$ such that if*

$$p \in \hat{\mathcal{C}}_i^{(\varepsilon_1, 0)} \quad \text{and} \quad q \in \hat{\mathcal{C}}_i^{(n\varepsilon_1, \varepsilon_2)},$$

*then* $\sigma \nmid \overline{pq}$.

*Proof.* Fix $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$. For each $n \geq 0$, consider the cycles

$$p \in \hat{\mathcal{C}}_i^{(\varepsilon_1,0)} \quad \text{and} \quad q_n \in \hat{\mathcal{C}}_i^{(n\varepsilon_1,\varepsilon_2)}.$$

(1) We claim that for each $n \geq 1$, $q_n^+$ lies in the region $\mathcal{R}_{p^2q_{n-1},q_{n+1}}$ (modulo $I$). This setup is shown in Figure 9.i. Indeed, suppose $q_n^+$ intersects $q_{n-1}^+$ as shown in Figure 9.ii. Then $q_{n-1}$ and $q_n$ factor into paths

$$q_{n-1} = s_3s_2s_1 \quad \text{and} \quad q_n = t_3t_2t_1,$$

where $\mathrm{t}(s_2^+) = \mathrm{t}(t_2^+)$ and $\mathrm{h}(s_2^+) = \mathrm{h}(t_2^+)$. In particular, there is some $m \in \mathbb{Z}$ such that

$$\bar{s}_2 = \bar{t}_2\sigma^m,$$

by Lemma 2.3.2. Set

$$r := t_3s_2t_1.$$

Since $q_{n-1}$ and $q_n$ are in $\hat{\mathcal{C}}$, we have

$$\sigma \nmid \bar{q}_{n-1} \quad \text{and} \quad \sigma \nmid \bar{q}_n, \tag{22}$$

by Proposition 2.20.1. Whence $\sigma \nmid \bar{s}_2$ and $\sigma \nmid \bar{t}_2$. Thus $m = 0$. Therefore

$$\bar{r} = \bar{t}_3\bar{s}_2\bar{t}_1 = \bar{t}_3\bar{t}_2\bar{t}_1 = \bar{q}_n.$$

In particular, $\sigma \nmid \bar{r}$ by (22). Thus the cycle $r$ is in $\hat{\mathcal{C}}$ by Lemmas 2.8.4 and 2.17. Furthermore, $r$ is in $\mathcal{C}_i^{(n\varepsilon_1,\varepsilon_2)}$ by construction. Whence $r$ is in $\hat{\mathcal{C}}_i^{(n\varepsilon_1,\varepsilon_2)}$. Therefore $r = q_n$ (modulo $I$) by Proposition 2.20.4. This proves our claim.

(2) By Claim (1), there is a cycle $r \in \hat{\mathcal{C}}^{(\varepsilon_1,0)}$ such that the area of the region

$$\mathcal{R}_{s_nq'_n,q'_{n+1}}$$

bounded by a rightmost subpath ${q'}_n^+$ of $q_n^+$, a rightmost subpath ${q'}_{n+1}^+$ of $q_{n+1}^+$, and a subpath $s_n^+$ of a path in $\pi^{-1}(r^\infty)$, tends to zero (modulo $I$) as $n \to \infty$. See Figure 9.iii. (The case $r = p$ is shown in Figure 9.i.) Since $Q$ is finite, there is some $N \gg 1$ such that if $n \geq N$, then

$$q'_{n+1} = s_nq'_n \quad (\text{modulo } I). \tag{23}$$

Now choose $m \geq 1$ sufficiently large so that $r$ is a subpath of

$$s := s_{N+m-1} \cdots s_{N+1}s_N.$$

By iterating (23) $m-1$ times, we obtain

$$q'_{N+m} = sq'_N.$$

Furthermore, there is a simple matching $D \in \mathcal{S}$ such that $x_D \nmid \bar{q}_{N+m}$, by Proposition 2.20.1. Whence

$$x_D \nmid \bar{q}'_{N+m} = \bar{s}\bar{q}'_N.$$

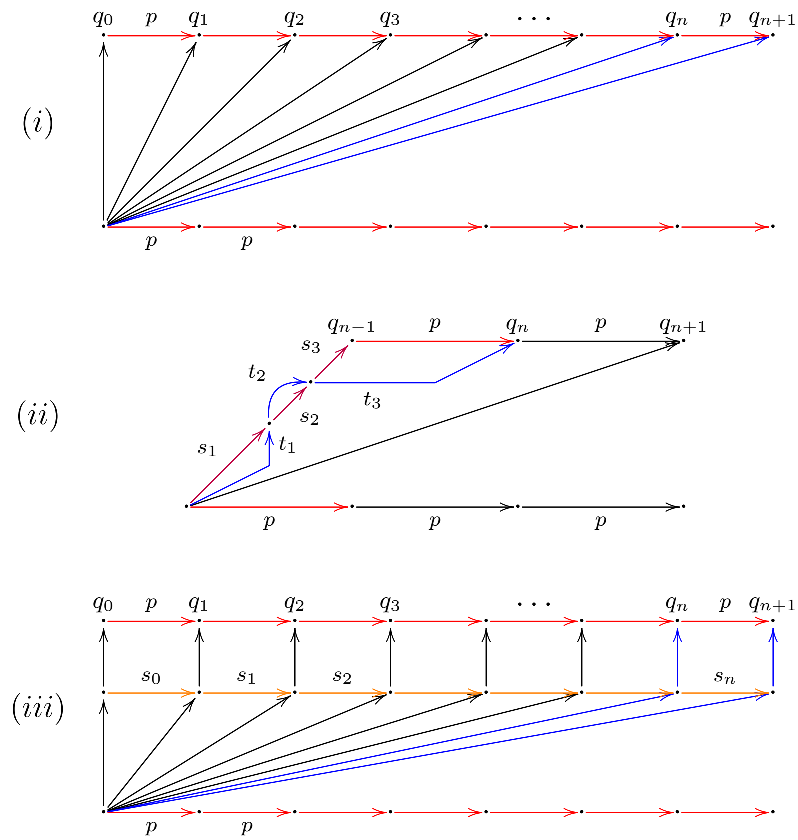


FIGURE 9. Setup for Lemma 2.21. In (ii): $q_{n-1}^+$ and $q_n^+$ intersect, and thus factor into paths $q_{n-1} = s_3 s_2 s_1$ and $q_n = t_3 t_2 t_1$, drawn in purple and blue respectively. In (iii): the area of the region $\mathcal{R}_{s_n q'_n, q'_{n+1}}$ tends to zero as $n \to \infty$. Here, an infinite path in $\pi^{-1}(r^\infty)$ is drawn in orange, each lift of the cycle $p$ is drawn in red, and $q_n^+$ and $q_{n+1}^+$ are drawn in blue.

In particular,

$$x_D \nmid \bar{s}. \tag{24}$$

Moreover, $\bar{r} \mid \bar{s}$ since $r$ is a subpath of $s$, by our choice of $m$. Whence $x_D \nmid \bar{r}$ by (24). But $\bar{p} = \bar{r}$ since $p$ and $r$ are both in $\hat{\mathcal{C}}^{(\varepsilon_1, 0)}$, by Proposition 2.20.2. Thus $x_D \nmid \bar{p}$. Whence $x_D \nmid \bar{p}\bar{q}_{N+m}$. Therefore

$$\sigma \nmid \bar{p}\bar{q}_{N+m}.$$

□

Consider the subset of arrows

$$Q_1^{\mathcal{S}} := \{a \in Q_1 \mid a \notin D \text{ for each } D \in \mathcal{S}\}, \tag{25}$$

where $\mathcal{S}$ is the set of simple matchings of $A$.

We will show in Theorem 2.24 below that the two assumptions considered in the following lemma, namely that $Q_1^{\mathcal{S}} \neq \emptyset$ and $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$, can never both hold.

**Lemma 2.22.** *Suppose $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$. Let $\delta \in Q_1^{\mathcal{S}}$. Then there is a cycle $p \in \hat{\mathcal{C}}_{\mathrm{t}(\delta)}$ such that $\delta$ is not a rightmost arrow subpath of $p$ (modulo $I$).*

*Proof.* Suppose $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$, and let $\delta \in Q_1^{\mathcal{S}}$. Assume to the contrary that there is an arrow $\delta$ which is a rightmost arrow subpath of each cycle in $\hat{\mathcal{C}}_{\mathrm{t}(\delta)}$.

(i) We first claim that there is some $u \in \mathbb{Z}^2 \setminus 0$ such that $\mathrm{t}(\delta^+)$ lies in the interior $\mathcal{R}_{s,t}^{\circ}$ of the region bounded by the lifts of representatives $s, t \in kQ$ of the cycle in $\hat{\mathcal{C}}^u_{\mathrm{h}(\delta)}$. (There is precisely one cycle in $\hat{\mathcal{C}}^u_{\mathrm{h}(\delta)}$ by Proposition 2.20.4.)

Indeed, by Lemma 2.21, there are vectors $u_0 = u_{n+1}, u_1, u_2, \ldots, u_n \in \mathbb{Z}^2 \setminus 0$ such that for each $0 \leq m \leq n$, the cycles

$$p_m = p_m' \delta \in \hat{\mathcal{C}}^{u_m}_{\mathrm{h}(\delta)}$$

satisfy

$$\sigma \nmid \overline{p_{m+1} p_m}. \tag{26}$$

Assume to the contrary that $\delta^+$ is not contained in each region

$$\mathcal{R}_{p_{m+1}p_m, p_m p_{m+1}}.$$

Since $p_m$ is in $\hat{\mathcal{C}}$, its lift $p_m^+$ does not have a cyclic subpath. In particular, the tail of $\delta^+$ is only a vertex subpath of $p_m^+$ at its tail. We therefore have the setup given in Figure 10.

Set $p := p_0$ and $q := p_n$. Then there is a leftmost subpath $\delta p'$ of $p$ and a rightmost subpath $q'$ of $q$ such that $(\delta p' q')^+$ is a cycle in $Q^+$. In particular,

$$\overline{\delta p' q'} = \sigma^{\ell}$$

for some $\ell \geq 1$, by Lemmas 2.8.1 and 2.17. Whence $\sigma \mid \overline{pq}$. But this is a contradiction to (26).

Thus there is some $0 \leq m \leq n$ such that $\delta^+$ lies in $\mathcal{R}_{p_{m+1}p_m, p_m p_{m+1}}$. Furthermore, since $\mathrm{t}(\delta^+)$ is not a vertex subpath of $p_m^+$ or $p_{m+1}^+$, $\mathrm{t}(\delta^+)$ lies in the interior of $\mathcal{R}_{p_{m+1}p_m, p_m p_{m+1}}$. The claim then follows by setting

$$s + I = p_{m+1} p_m \quad \text{ and } \quad t + I = p_m p_{m+1}.$$

(ii) Let $s$ and $t$ be as in Claim (i). In particular, $\bar{s} = \bar{t}$. Assume to the contrary that there is a simple matching $D \in \mathcal{S}$ for which

$$x_D \nmid \bar{s} = \bar{t}.$$

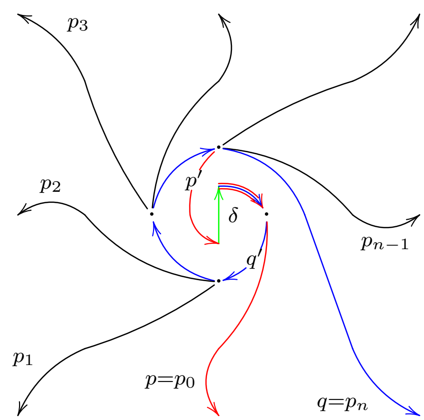


FIGURE 10. Setup for Claim (i) in the proof of Lemma 2.22. Here, $p, q \in \hat{\mathcal{C}}_{\mathrm{h}(\delta)}$ are drawn in red and blue respectively. Furthermore, $\delta p'$ is a leftmost subpath of $p$; $q'$ is a rightmost subpath of $q$; and $(\delta p' q')^+$ is a cycle in $Q^+$.

Let $r^+$ be a path in $\mathcal{R}_{s,t}$ from a vertex on the boundary of $\mathcal{R}_{s,t}$ to $\mathrm{t}(\delta^+)$. It suffices to suppose the tail of $r$ is a vertex subpath of $s$, in which case $s$ factors into paths

$$s = s_2 e_{\mathrm{t}(r)} s_1.$$

See Figure 11. Then $(rs_1\delta)^+$ is a cycle in $Q^+$. Whence

$$\overline{rs_1\delta} = \sigma^\ell$$

for some $\ell \geq 1$, by Lemmas 2.8.1 and 2.17. In particular,

$$x_D \mid \overline{rs_1\delta} = \bar{r}\bar{s}_1\bar{\delta}.$$

Furthermore, by assumption,

$$x_D \nmid \bar{s}_1 \quad \text{and} \quad x_D \nmid \bar{\delta}.$$

Whence $x_D \mid \bar{r}$. Thus, since $D \in \mathcal{S}$ and $r$ are arbitrary, $\sigma$ divides the $\bar{\tau}$-image of each path in $\mathcal{R}_{s,t}$ from the boundary of $\mathcal{R}_{s,t}$ to $\mathrm{t}(\delta^+)$. But $\mathrm{t}(\delta^+)$ lies in the interior $\mathcal{R}^\circ_{s,t}$. Thus the vertex $\mathrm{t}(\delta)$ is a source in $Q \setminus D$. Therefore $D$ is not simple, contrary to assumption. It follows that $x_D \mid \bar{s}$ for each $D \in \mathcal{S}$.

(iii) By Claim (ii), $\sigma \mid \bar{s}$. But this is a contradiction since $s$ is in $\hat{\mathcal{C}}$, by Lemmas 2.8.3 and 2.17. Therefore there is a cycle in $\hat{\mathcal{C}}_{\mathrm{t}(\delta)}$ whose rightmost arrow subpath is not $\delta$ (modulo $I$). □

**Remark 2.23.** There are dimer algebras with an arrow $\delta \in Q_1$ that is a rightmost arrow subpath of each cycle in $\hat{\mathcal{C}}_{\mathrm{t}(\delta)}$; see Figure 12. Furthermore, if $A$ has center $Z$, admits a cyclic contraction (Definition 4.3), and $\delta \in Q_1^{\mathcal{S}}$, then $\delta$ is a rightmost arrow subpath of each cycle $p \in Ze_{\mathrm{t}(\delta)}$ for which $\sigma \nmid \bar{p}$, by [B3, Lemma 2.4].

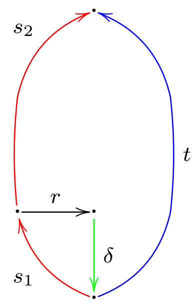


FIGURE 11. Setup for Claim (ii) in the proof of Lemma 2.22.

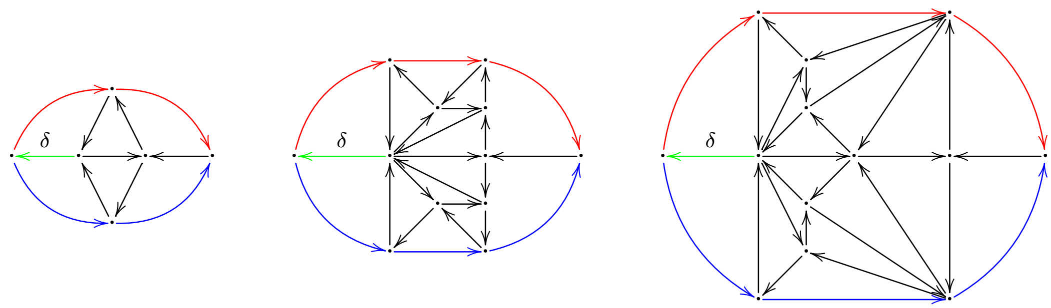


FIGURE 12. In each example, the arrow $\delta$ is a rightmost arrow subpath of each path from $\mathrm{t}(\delta)$ to a vertex on the boundary of the region $\mathcal{R}_{p,q}$ bounded by the paths $p^+$ and $q^+$, drawn in red and blue respectively. The red and blue arrows are non-vertex paths in $Q^+$, and the black arrows are arrows in $Q^+$.

The following will be generalized in Theorem 4.41 below.

**Theorem 2.24.** *Suppose (i) $A$ is cancellative, or (ii) $\hat{\mathcal{C}}^u_i \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$. Then $Q_1^{\mathcal{S}} = \emptyset$, that is, each arrow annihilates a simple $A$-module of dimension $1^{Q_0}$.*

*Proof.* Recall that assumption (i) implies assumption (ii), by Proposition 2.10. So suppose (ii) holds, and assume to the contrary that there is an arrow $\delta$ in $Q_1^{\mathcal{S}}$.

By Lemma 2.22, there is a cycle $p \in \hat{\mathcal{C}}_{\mathrm{t}(\delta)}$ whose rightmost arrow subpath is not $\delta$ (modulo $I$). Let $u \in \mathbb{Z}^2$ be such that $p \in \mathcal{C}^u$. By assumption, there is a cycle $q$ in $\hat{\mathcal{C}}^u_{\mathrm{h}(\delta)}$. We thus have one of the three cases given in Figure 13.

First suppose $p^+$ and $q^+$ do not intersect (that is, do not share a common vertex), as shown in case (i). Then $p^+$ and $q^+$ bound a column. By Lemma 2.15, the brown arrows belong to a simple matching $D \in \mathcal{S}$. In particular, $\delta$ is in $D$, contrary to assumption.

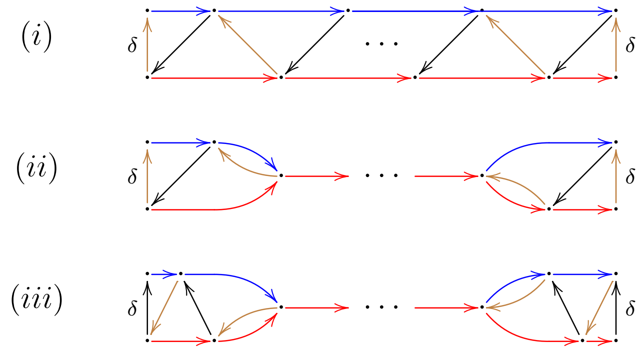


FIGURE 13. Setup for Theorem 2.24. The red and blue paths are the (lifts of the) cycles $p$ and $q$, respectively. The red and blue arrows are paths in $Q^+$, and the black and brown arrows are arrows in $Q^+$. In cases (i) and (ii), $\delta$ belongs to a simple matching, contrary to assumption. In case (iii), $\delta$ is a rightmost arrow subpath of $p$ (modulo $I$), again contrary to assumption.

So suppose $p^+$ and $q^+$ intersect, as shown in cases (ii) and (iii). Then $p^+$ and $q^+$ bound a union of pillars. Again by Lemma 2.15, the brown arrows belong to a simple matching $D \in \mathcal{S}$. In particular, in case (ii) $\delta$ is in $D$, contrary to assumption. Therefore case (iii) holds. But then $\delta$ is a rightmost arrow subpath of $p$ (modulo $I$), contrary to our choice of $p$. □

## 3. Cancellative dimer algebras

In this section, we give an explicit impression of all cancellative dimer algebras. Throughout, $A = kQ/I$ is a cancellative dimer algebra, and we use the notation (6).

**Lemma 3.1.** *Let $p, q \in e_jAe_i$ be paths such that*

$$\mathrm{t}(p^+) = \mathrm{t}(q^+) \quad \textit{and} \quad \mathrm{h}(p^+) = \mathrm{h}(q^+).$$

*If $\bar{p} = \bar{q}$, then $p = q$.*

*Proof.* Since $A$ is cancellative, $A$ has at least one simple matching by Lemma 2.16. In particular, $\sigma \neq 1$. Thus we may apply the proof of Lemma 2.3.3, with $\bar{\tau}$ in place of $\bar{\eta}$. □

**Lemma 3.2.** *For each $i \in Q_0$, the corner ring $e_iAe_i$ is generated by $\sigma_i$ and $\hat{\mathcal{C}}_i$.*

*Proof.* Since $I$ is generated by binomials, $e_iAe_i$ is generated by $\mathcal{C}_i$. It thus suffices to show that $\mathcal{C}_i$ is generated by $\sigma_i$ and $\hat{\mathcal{C}}_i$.

Let $u \in \mathbb{Z}^2$ and $p \in \mathcal{C}_i^u$. If $u = 0$, then $p = \sigma_i^m$ for some $m \geq 0$ by Lemma 2.8.2. So suppose $u \neq 0$. Then there is a cycle $q$ in $\hat{\mathcal{C}}_i^u$ by Proposition 2.10. In particular,

$\bar{p} = \bar{q}\sigma^m$ for some $m \in \mathbb{Z}$ by Lemma 2.18. Furthermore, $m \geq 0$ by Proposition 2.20.1. Therefore $p = q\sigma_i^m$ by Lemma 3.1. □

It is well known that if $A$ is cancellative, then $A$ is a 3-Calabi-Yau algebra [D, MR]. In particular, the center $Z$ of $A$ is noetherian and reduced, and $A$ is a finitely generated $Z$-module. In the following, we give independent proofs of these facts so that Theorem 3.5 may be self-contained. Furthermore, we prove the converse for dimer algebras which admit cyclic contractions in Theorem 4.50 below.

**Theorem 3.3.** *Suppose $A$ is a cancellative dimer algebra, and let $i, j \in Q_0$. Then*

*(1) $e_iAe_i = Ze_i \cong Z$.*
*(2) $\bar{\tau}(e_iAe_i) = \bar{\tau}(e_jAe_j)$.*
*(3) $A$ is a finitely generated $Z$-module, and $Z$ is a finitely generated $k$-algebra.*
*(4) $Z$ is reduced.*

*Proof.* (1) For each $i \in Q_0$ and $u \in \mathbb{Z}^2 \setminus 0$, there exists a unique cycle $c_{ui} \in \hat{\mathcal{C}}_i^u$ (modulo $I$) by Propositions 2.20.4 and 2.10. Thus the sum

$$\sum_{i \in Q_0} c_{ui} \in \bigoplus_{i \in Q_0} e_iAe_i$$

is in $Z$, by Lemma 2.19.1. Whence $e_iAe_i \subseteq Ze_i$ by Lemma 3.2. Furthermore,

$$Ze_i = Ze_i^2 = e_iZe_i \subseteq e_iAe_i.$$

Therefore $Ze_i = e_iAe_i$.

We now claim that there is an algebra isomorphism $Z \cong Ze_i$ for each $i \in Q_0$. Indeed, fix $i \in Q_0$ and suppose $z \in Z$ is nonzero. Then there is some $j \in Q_0$ such that $ze_j \neq 0$. Furthermore, since $Q$ is a dimer quiver, there is a path $p$ from $i$ to $j$.

Assume to the contrary that $ze_jp = 0$. Thus, since $I$ is generated by binomials, it suffices to suppose $ze_j = c_1 - c_2$ where $c_1$ and $c_2$ are paths. But since $A$ is cancellative, $ze_jp = 0$ implies $c_1 = c_2$. Whence $ze_j = 0$, a contradiction. Therefore $ze_jp \neq 0$. Consequently,

$$pe_iz = pz = zp = ze_jp \neq 0.$$

Whence $ze_i \neq 0$. Thus the algebra homomorphism $Z \to Ze_i$, $z \mapsto ze_i$, is injective, hence an isomorphism. This proves our claim.

(2) Follows from Proposition 2.20.2 and and Lemma 3.2.

(3) $A$ is generated as a $Z$-module by all paths of length at most $|Q_0|$ by Claim (1) and [B, second paragraph of proof of Theorem 2.11]. Thus $A$ is a finitely generated $Z$-module. Furthermore, $A$ is a finitely generated $k$-algebra since $Q$ is finite. Therefore $Z$ is also a finitely generated $k$-algebra [McR, 1.1.3].

(4) Suppose $z \in Z$ and $z^n = 0$. Fix $i \in Q_0$. Then

$$(ze_i)^n = z^n e_i = 0.$$

Furthermore, we may write

$$ze_i = \sum_{j=1}^{\ell} p_j \in e_i A e_i,$$

where each $p_j$ is a cycle with a scalar coefficient. For $1 \leq j \leq \ell$, let $u_j \in \mathbb{Z}^2$ be such that $p_j \in \mathcal{C}^{u_j}$. Then the path summand $p_{j_n} \cdots p_{j_1}$ of $(ze_i)^n$ is in $\mathcal{C}^{\sum_{k=1}^{n} u_{j_k}}$. In particular, if $u_k$, $1 \leq k \leq n$, has maximal length, then $p_k^n$ is the only path summand of $(ze_i)^n$ which is in $\mathcal{C}^{nu_k}$. But no path is zero modulo $I$, and so $z = 0$. Therefore $Z$ is reduced. □

**Lemma 3.4.** *Let $p \in \mathcal{C}^u$. Then $u = 0$ if and only if $\bar{p} = \sigma^m$ for some $m \geq 0$.*

*Proof.* $(\Rightarrow)$ Lemma 2.8.1.

$(\Leftarrow)$ Let $u \in \mathbb{Z}^2 \setminus 0$, and assume to the contrary that $p \in \mathcal{C}^u$ satisfies $\bar{p} = \sigma^m$ for some $m \geq 0$. Since $A$ is cancellative, there is a cycle $q \in \hat{\mathcal{C}}^u_{\mathrm{t}(p)}$ by Proposition 2.10. Furthermore, $\sigma \nmid \bar{q}$ by Proposition 2.20.1. Thus there is some $n \geq 0$ such that

$$\bar{p} = \bar{q}\sigma^n,$$

by Lemma 2.3.2. Whence $\sigma^m = \bar{q}\sigma^n$. Furthermore, $\sigma \neq 1$ by Lemma 2.16. Therefore $m = n$ and

$$\bar{q} = 1, \tag{27}$$

since $B$ is a polynomial ring. But each arrow in $Q$ is contained in a simple matching by Theorem 2.24. Therefore $\bar{q} \neq 1$, contrary to (27). □

We now prove our main theorem for this section.

**Theorem 3.5.** *Suppose $A = kQ/I$ is a cancellative dimer algebra. Then the algebra homomorphism*

$$\tau : A \to M_{|Q_0|}(B),$$

*defined in (4), is an impression of $A$. Therefore $\tau$ classifies all simple $A$-module isoclasses of maximal $k$-dimension. Furthermore, the following holds:*

$$Z \cong k\left[\cap_{i \in Q_0} \bar{\tau}\left(e_i A e_i\right)\right] = k\left[\cup_{i \in Q_0} \bar{\tau}\left(e_i A e_i\right)\right]. \tag{28}$$

*Proof.* (i) We first show that $\tau$ is injective.

(i.a) We claim that $\tau$ is injective on the vertex corner rings $e_i A e_i$, $i \in Q_0$. Fix a vertex $i \in Q_0$ and let $p, q \in e_i A e_i$ be cycles satisfying $\bar{p} = \bar{q}$. Let $r$ be a path such that $r^+$ is path from $\mathrm{h}(p^+)$ to $\mathrm{t}(p^+)$. Then $rp \in \mathcal{C}^0$. Thus there is some $m \geq 0$ such that

$$rp = \sigma_i^m,$$

by Lemma 2.8.2. Whence

$$\overline{rq} = \bar{r}\bar{q} = \bar{r}\bar{p} = \overline{rp} = \overline{\sigma_i^m} = \bar{\sigma}_i^m = \sigma^m.$$

Thus $rq \in \mathcal{C}^0$ by Lemma 3.4. Hence $rq = \sigma_i^m = rp$ by Lemma 3.1. Therefore $p = q$ since $A$ is cancellative.

(i.b) We now claim that $\tau$ is injective on paths. Let $p, q \in e_j A e_i$ be paths satisfying $\bar{p} = \bar{q}$. Let $r$ be a path from $j$ to $i$. The two cycles $pr$ and $qr$ at $j$ then satisfy $\overline{pr} = \overline{qr}$. Thus $pr = qr$ since $\tau$ is injective on the corner ring $e_j A e_j$ by Claim (i.a). Therefore $p = q$ since $A$ is cancellative.

Since $A$ is generated by paths and $\tau$ is injective on paths, it follows that $\tau$ is injective.

(ii) For each

$$\mathfrak{b} \in \mathcal{Z}(\sigma)^c \subset \operatorname{Max} B = \mathbb{A}_k^{|\mathcal{S}|},$$

the composition $\epsilon_{\mathfrak{b}}\tau$ defined in (8) is a representation of $A$ where, when viewed as a vector space diagram on $Q$, each arrow is represented by a non-zero scalar. Thus $\epsilon_{\mathfrak{b}}\tau$ is simple since there is an oriented path between any two vertices in $Q$. Therefore $\epsilon_{\mathfrak{b}}\tau$ is surjective.

(iii) We claim that the morphism

$$\operatorname{Max} B \to \operatorname{Max} \tau(Z), \quad \mathfrak{b} \mapsto \mathfrak{b}1_{|Q_0|} \cap \tau(Z),$$

is surjective. Indeed, for any $\mathfrak{m} \in \operatorname{Max} R$, $B\mathfrak{m}$ is a (nonzero) proper ideal of $B$. Thus there is a maximal ideal $\mathfrak{b} \in \operatorname{Max} B$ containing $B\mathfrak{m}$ since $B$ is noetherian. Furthermore, since $B$ is a finitely generated $k$-algebra and $k$ is algebraically closed, the intersection $\mathfrak{b} \cap R =: \mathfrak{m}'$ is a maximal ideal of $R$. Whence

$$\mathfrak{m} \subseteq B\mathfrak{m} \cap R \subseteq \mathfrak{b} \cap R = \mathfrak{m}'.$$

But $\mathfrak{m}$ and $\mathfrak{m}'$ are both maximal ideals of $R$. Thus $\mathfrak{m} = \mathfrak{m}'$. Therefore $\mathfrak{b} \cap R = \mathfrak{m}$, proving our claim.

Claims (i), (ii), (iii) imply that $(\tau, B)$ is an impression of $A$.

(iv) Since $(\tau, B)$ is an impression of $A$, $Z$ is isomorphic to $R$ by [B, Lemma 2.1 (2)]. Furthermore, $R$ is equal to $S$ by Theorem 3.3.2. Therefore (28) holds. □

**Remark 3.6.** The labeling of arrows we obtain, namely (4), agrees with the labeling of arrows in the toric construction of [CQ, Proposition 5.3]. We note, however, that impressions are defined more generally for non-toric algebras and have different implications than the toric construction of [CQ].

**Corollary 3.7.** *A dimer algebra $A$ is cancellative if and only if it admits an impression $(\tau, B)$ where $B$ is an integral domain and $\tau(e_i) = E_{ii}$ for each $i \in Q_0$.*

*Proof.* Suppose $A$ admits an impression $(\tau, B)$ such that $B$ is an integral domain and $\tau(e_i) = E_{ii}$ for each $i \in Q_0$. Consider paths $p, q, r$ satisfying $pr = qr \neq 0$. Then

$$\bar{p}\bar{r} = \overline{pr} = \overline{qr} = \bar{q}\bar{r}.$$

Thus $\bar{p} = \bar{q}$ since $B$ is an integral domain. Whence

$$\tau(p) = \bar{p}E_{\mathrm{h}(p),\mathrm{h}(r)} = \bar{q}E_{\mathrm{h}(p),\mathrm{h}(r)} = \tau(q).$$

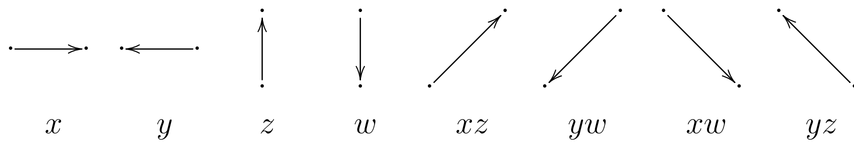


FIGURE 14. A labeling of arrows in the quiver of a square dimer algebra that specifies an impression.

Therefore $p = q$ by the injectivity of $\tau$.

The converse follows from Theorem 3.5. $\square$

**Remark 3.8.** Although non-cancellative dimer algebras do not admit impressions $(\tau, B)$ with $B$ an integral domain by Corollary 3.7, we will find that their homotopy algebras do; see Definition 4.33 and Theorem 4.35.

**Example 3.9.** A dimer algebra $A = kQ/I$ is *square* if the underlying graph of its cover $Q^+$ is a square grid graph with vertex set $\mathbb{Z} \times \mathbb{Z}$, and with at most one diagonal edge in each unit square. Examples of square dimer algebras are given in Figure 15 (i.b, ii.b, iii.c, iv.c).

By [B, Theorem 3.7], any square dimer algebra $A$ admits an impression $(\tau, B = k[x, y, z, w])$, where for each arrow $a \in Q_1^+$, $\bar{\tau}(a)$ is the monomial corresponding to the orientation of $a$ given in Figure 14. Specifically, $\tau : A \to M_{|Q_0|}(B)$ is the algebra homomorphism defined by

$$\tau(e_i) = E_{ii} \quad \text{and} \quad \tau(a) = \bar{\tau}(a) E_{\mathrm{h}(a),\mathrm{t}(a)}$$

for each $i \in Q_0$ and $a \in Q_1$. If $Q$ only possesses three arrow orientations, say up, left, and right-down, then we may label the respective arrows by $x$, $y$, and $z$, and obtain an impression $(\tau, k[x, y, z])$. In either case, $A$ is cancellative by Corollary 3.7.

**Proposition 3.10.** *Suppose $A$ is cancellative, and let $p \in \mathcal{C}$ be a non-vertex cycle. Then $p \in \hat{\mathcal{C}}$ if and only if $\sigma \nmid \bar{p}$. In particular, $\tau(Z) \subseteq B$ is generated over $k$ by $\sigma$ and a set of monomials in $B$ not divisible by $\sigma$.*

*Proof.* ($\Rightarrow$) Proposition 2.20.1.

($\Leftarrow$) Lemma 2.8.3. $\square$

Recall that an ideal $\mathfrak{p}$ in a (possibly noncommutative) ring $R$ is prime if for all ideals $I, J \lhd R$, $IJ \subseteq \mathfrak{p}$ implies $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$. Moreover, $R$ is prime if for all $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, that is, the zero ideal is a prime ideal.

**Proposition 3.11.** *Cancellative dimer algebras are prime.*

*Proof.* We claim that for nonzero elements $p, q \in A$, we have

$$qAp \neq 0.$$

It suffices to suppose that

$$p \in e_j A e_i \quad \text{and} \quad q \in e_\ell A e_k.$$

Since $Q$ is strongly connected, there is a path $r$ from $j$ to $k$. Furthermore, the polynomials $\bar{p}$, $\bar{q}$, and $\bar{r}$ are nonzero since $\tau$ is injective, by Theorem 3.5. Thus the product $\overline{qrp} = \bar{q}\bar{r}\bar{p} \in B$ is nonzero since $B$ is an integral domain. Therefore $qrp$ is nonzero. □

## 4. Non-cancellative dimer algebras and their homotopy algebras

Throughout, we use the notation (7).

### 4.1. Cyclic contractions.

In this section, we introduce a new method for studying non-cancellative dimer algebras that is based on the notion of Higgsing, or more generally symmetry breaking, in physics. Using this strategy, we gain information about non-cancellative dimer algebras by relating them to cancellative dimer algebras with certain similar structure. Throughout, $A = kQ/I$ is a dimer algebra, usually non-cancellative.

In the following, we introduce a $k$-linear map induced from the operation of edge contraction in graph theory.

**Definition 4.1.** Let $Q = (Q_0, Q_1, \mathrm{t}, \mathrm{h})$ be a dimer quiver, let $Q_1^* \subset Q_1$ be a subset of arrows, and let $Q' = (Q_0', Q_1', \mathrm{t}', \mathrm{h}')$ be the quiver obtained by contracting each arrow in $Q_1^*$ to a vertex. Specifically,

$$Q_0' := Q_0 / \{\mathrm{h}(a) \sim \mathrm{t}(a) \mid a \in Q_1^*\}, \quad Q_1' = Q_1 \setminus Q_1^*,$$

and $\mathrm{h}'(a) = \mathrm{h}(a)$, $\mathrm{t}'(a) = \mathrm{t}(a)$ for each $a \in Q_1'$. Then there is a $k$-linear map of path algebras

$$\psi : kQ \to kQ'$$

defined by

$$\psi(a) = \begin{cases} a & \text{if } \ a \in Q_0 \cup Q_1 \setminus Q_1^* \\ e_{\mathrm{t}(a)} & \text{if } \ a \ \in Q_1^* \end{cases}$$

and extended multiplicatively to nonzero paths and $k$-linearly to $kQ$. If $\psi$ induces a $k$-linear map of dimer algebras

$$\psi : A = kQ/I \to A' = kQ'/I',$$

that is, $\psi(I) \subseteq I'$, then we call $\psi$ a *contraction of dimer algebras.*

**Remark 4.2.** The containment $\psi(I) \subseteq I'$ may be proper. Indeed, $\psi(I) \neq I'$ whenever $\psi$ contracts a unit cycle to a removable 2-cycle.

We now describe the structure we wish to preserve under a contraction. To specify this structure, we introduce the following commutative algebras.

**Definition 4.3.** Let $\psi : A \to A'$ be a contraction to a cancellative dimer algebra, and let $(\tau, B)$ be an impression of $A'$. If

$$S := k\left[\cup_{i\in Q_0}\bar{\tau}\psi\left(e_iAe_i\right)\right] = k\left[\cup_{i\in Q_0'}\bar{\tau}\left(e_iA'e_i\right)\right] =: S',$$

then we say $\psi$ is *cyclic*, and call $S$ the *cycle algebra* of $A$.

The cycle algebra is independent of the choice of $\psi$ and $\tau$ by [B3, Theorem 3.13]. Henceforth we will consider cyclic contractions $\psi : A \to A'$.

**Notation 4.4.** For $p \in e_jAe_i$ and $q \in e_\ell A'e_k$, set

$$\bar{p} := \bar{\tau}\psi(p) \in B \quad \text{ and } \quad \bar{q} := \bar{\tau}(q) \in B.$$

An immediate question is whether all non-cancellative dimer algebras admit cyclic contractions. We will show that dimer algebras exist which do not admit contractions (cyclic or not) to cancellative dimer algebras. For example, dimer algebras that contain permanent 2-cycles cannot be contracted to cancellative dimer algebras (Proposition 4.43). Furthermore, we will show that if $\psi : A \to A'$ is a cyclic contraction and $A$ is cancellative, then $\psi$ is necessarily trivial (Theorem 4.41).

**Example 4.5.** Consider the three examples of contractions $\psi : A \to A'$ given in Figure 15. In each example, $\psi$ is cyclic; $Q'$ is a square dimer (Example 3.9) with an impression given by the arrow orientations in Figure 14; and $B = k[x, y, z, w]$. In particular, quiver (a) is non-cancellative, quivers (b) and (c) are cancellative, and quiver (c) is obtained by deleting the removable 2-cycles in quiver (b). The non-cancellative quivers (a) first appeared respectively in [FHPR, Section 4], [FKR]; [Bo, Example 3.2]; and [DHP, Table 6, 2.6]. Their respective cycle algebras $S$ are:[6]

$$\begin{aligned}
&\text{(i):} && S = k\left[xz, xw, yz, yw\right] && R = k + (x^2zw, xyzw, y^2zw)S\\
&\text{(ii):} && S = k\left[xz, yz, xw, yw\right] && R = k + (xz, yz)S\\
&\text{(iii):} && S = k\left[xz, yw, x^2w^2, y^2z^2\right] && R = k + (yw, x^2w^2, y^2z^2)S
\end{aligned}$$

**Notation 4.6.** For $g, h \in B$, by $g \mid h$ we mean $g$ divides $h$ in $B$, even if $g$ or $h$ is assumed to be in $S$.

**Lemma 4.7.** *Suppose $\psi : A \to A'$ is a contraction of dimer algebras, and $A'$ has a perfect matching. Then $\psi$ cannot contract a unit cycle of $A$ to a vertex.*

*Proof.* Assume to the contrary that $\psi$ contracts the unit cycle $\sigma_j \in A$ to the vertex $e_{\psi(j)} \in A'$. Fix a unit cycle $\sigma_{i'} \in A'$. Since $\psi$ is surjective on $Q_0'$, there is a vertex $i \in Q_0$ such that $\psi(i) = i'$. Let $p \in A$ be a path from $i$ to $j$, and set $p' := \psi(p)$. Then

$$p'\sigma_{i'} \overset{\text{(I)}}{=} \psi(p\sigma_i) \overset{\text{(II)}}{=} \psi(\sigma_j p) \overset{\text{(III)}}{=} \psi(\sigma_j)p' = e_{\psi(j)}p' = p'. \tag{29}$$

Indeed, (I) and (III) hold by Definition 4.1, and (II) holds by Lemma 1.5.

[6]The algebra $R$ will be introduced in Definition 4.26 below.

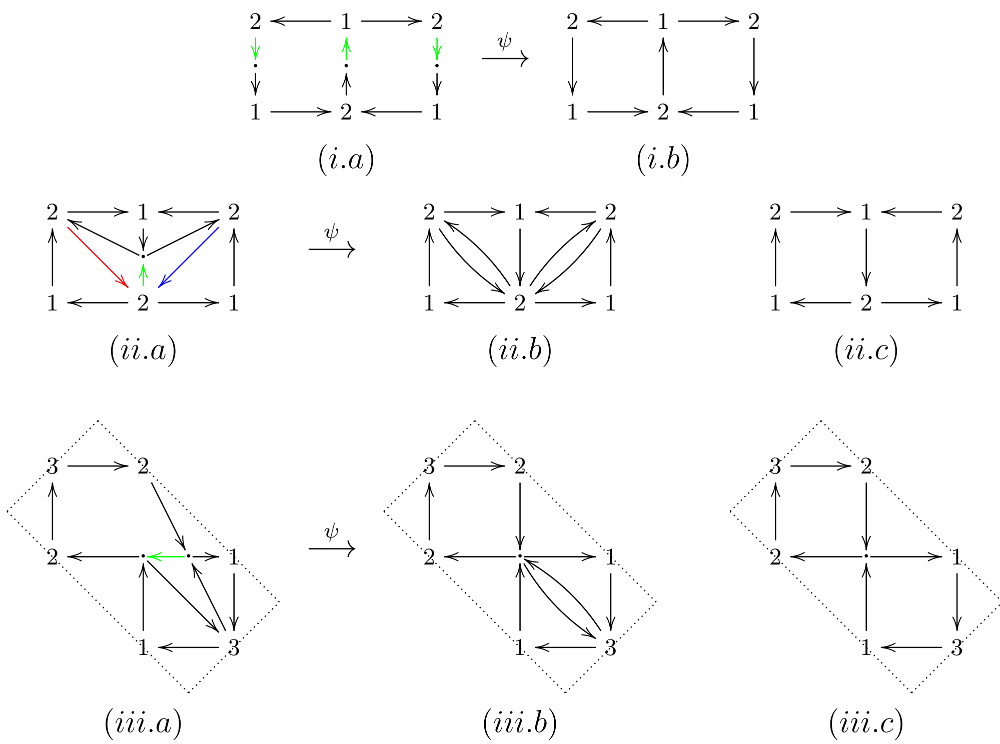


FIGURE 15. Some non-cancellative dimer algebras that cyclically contract to square dimer algebras. Each quiver is drawn on a torus, and the contracted arrows are drawn in green. In (ii.a), the red and blue arrows generate a free subalgebra of $A$; see Example 4.48 below.

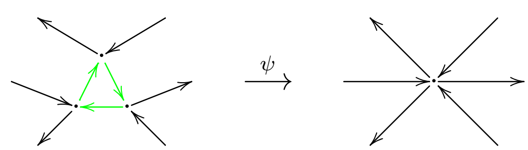


FIGURE 16. The contraction of a unit cycle (drawn in green) to a vertex. Such a contraction cannot induce a contraction of dimer algebras $\psi : A \to A'$ if $A'$ has a perfect matching, by Lemma 4.7.

Denote by $\mathcal{P}'$ is the set of perfect matchings of $A'$. Set $\sigma := \prod_{D \in \mathcal{P}'} x_D$. Then (29) implies

$$\bar{\eta}(p')\sigma = \bar{\eta}(p'\sigma_{i'}) = \bar{\eta}(p') \in k\left[x_D \mid D \in \mathcal{P}'\right],$$

by Lemma 1.6. Whence $\sigma = 1$. But this contradicts our assumption that $\mathcal{P}' \neq \emptyset$. $\square$

An example where a unit cycle is contracted to a vertex is given in Figure 16.

**Lemma 4.8.** *Suppose $\psi : A \to A'$ is a contraction of dimer algebras, and $A'$ has a perfect matching. Then $\psi$ cannot contract a non-vertex cycle in the underlying graph $\overline{Q}$ of $Q$ to a vertex. In particular,*

*(1) $\psi$ cannot contract a non-vertex cycle in $Q$ to a vertex;*
*(2) if $p \in \mathcal{C} \setminus \mathcal{C}^0$, then $\psi(p) \in \mathcal{C}' \setminus \mathcal{C}'^0$; and*
*(3) $A$ does not have a non-cancellative pair where one of the paths is a vertex.*

*Proof.* The number of vertices, edges, and faces in the underlying graphs $\overline{Q}$ and $\overline{Q}'$ of $Q$ and $Q'$ are given by

$$\begin{array}{lll} V = |Q_0|, & E = |Q_1|, & F = \# \text{ of connected components of } T^2 \setminus \overline{Q}, \\ V' = |Q'_0|, & E' = |Q'_1|, & F' = \# \text{ of connected components of } T^2 \setminus \overline{Q}'. \end{array}$$

Since $\overline{Q}$ and $\overline{Q}'$ each embed into a two-torus, their respective Euler characteristics vanish:

$$V - E + F = 0, \qquad V' - E' + F' = 0. \tag{30}$$

Assume to the contrary that $\psi$ contracts the cycles $p_1, \ldots, p_\ell$ in $\overline{Q}$ to vertices in $\overline{Q}'$. Denote by $n_0$ and $n_1$ the respective number of vertices and arrows in $Q$ which are subpaths of some $p_i$, $1 \leq i \leq \ell$. Denote by $m$ the number of vertices in $Q'_0$ of the form $\psi(p_i)$ for some $1 \leq i \leq \ell$. By assumption, $m \geq 1$.

In any cycle, the number of vertex subpaths equals the number of arrow subpaths. Furthermore, if two cycles share a common edge, then they also share a common vertex. Therefore

$$n_1 \geq n_0.$$

Whence

$$\begin{array}{rcl} 0 & = & F' - E' + V' \\ & = & F' - (E - n_1) + (V - n_0 + m) \\ & = & F' + (-E + V) + (n_1 - n_0) + m \\ & \geq & F' - F + m. \end{array}$$

Thus, since $m \geq 1$,

$$F' < F.$$

Therefore $\psi$ contracts a face of $Q$ to a vertex. In particular, some unit cycle in $Q$ is contracted to a vertex. But this is a contradiction by Lemma 4.7. □

**Remark 4.9.** Suppose $\psi : A \to A'$ is a cyclic contraction. Then $A'$ is cancellative by definition, and thus has a perfect matching by Lemma 2.16. Therefore Lemmas 4.7 and 4.8 hold in the case $\psi$ is cyclic.

**Remark 4.10.** Let $\psi : A \to A'$ be a contraction. Consider a path $p = a_n \cdots a_2 a_1 \in kQ$ with $a_1, \ldots, a_n \in Q_0 \cup Q_1$. If $p \neq 0$, then by definition

$$\psi(p) = \psi(a_n) \cdots \psi(a_1) \in kQ'.$$

However, we claim that if $\psi$ is non-trivial, then it is not an algebra homomorphism. Indeed, let $\delta \in Q_1^*$. Consider a path $a_2 \delta a_1 \neq 0$ in $A$. By Lemma 4.8.1, $\delta$ is not a cycle. In particular, $\mathrm{h}(a_1) \neq \mathrm{t}(a_2)$. Whence

$$\psi(a_2 a_1) = \psi(0) = 0 \neq \psi(a_2)\psi(a_1),$$

proving our claim. We note, however, that the restriction

$$\psi : \epsilon_0 A \epsilon_0 \to A' \quad \text{where } \ \epsilon_0 := 1_A - \sum_{\delta \in Q_1^*} e_{\mathrm{h}(\delta)},$$

is an algebra homomorphism [B3, Proposition 2.13.1].

**Lemma 4.11.** *Let $p$ be a non-vertex cycle.*

*(1) If $p \in \mathcal{C}^0$, then $\bar{p} = \sigma^m$ for some $m \geq 1$.*
*(2) If $p \in \mathcal{C} \setminus \hat{\mathcal{C}}$, then $\sigma \mid \bar{p}$.*

*Proof.* If $p^+$ is a cycle (resp. has a cyclic subpath) in $Q^+$, then $\psi(p)^+$ is a cycle (resp. has a cyclic subpath) in $Q'^+$. Furthermore, $A'$ is cancellative. Claims (1) and (2) therefore hold by Lemmas 2.8.1 and 2.8.3 respectively. □

The following strengthens Lemma 2.3.3 for dimer algebras that admit cyclic contractions (specifically, the head and tail of the lifts $p^+$ and $q^+$ are not required to coincide).

**Lemma 4.12.** *Let $p, q \in e_j A e_i$ be distinct paths. Then the following are equivalent:*

*(1) $\psi(p) = \psi(q)$.*
*(2) $\bar{p} = \bar{q}$.*
*(3) $p, q$ is a non-cancellative pair.*

*Proof.* (1) $\Rightarrow$ (3): Suppose $\psi(p) = \psi(q)$. Consider lifts $p^+$ and $q^+$ for which

$$\mathrm{t}(p^+) = \mathrm{t}(q^+).$$

Let $r^+$ be a path from $\mathrm{h}(p^+)$ to $\mathrm{h}(q^+)$. Then $\psi(r^+)$ is a cycle in $Q'^+$ since $\psi(p) = \psi(q)$, by Lemma 2.4.1. Thus $r^+$ is also a cycle by the contrapositive of Lemma 4.8.2. Whence $\mathrm{h}(p^+) = \mathrm{t}(q^+)$. Furthermore, $\sigma \neq 1$ since $A$ is cancellative, by Lemma 2.16. Therefore $p, q$ is a non-cancellative pair, by Claim (3.i) in the proof of Lemma 2.3 (with $\bar{\tau}\psi$ in place of $\bar{\eta}$).

(3) $\Rightarrow$ (2): Holds similar to Claim (3.ii) in the proof of Lemma 2.3.

(2) $\Rightarrow$ (1): Holds since $\bar{\tau} : e_{\psi(j)} A' e_{\psi(i)} \to B$ is injective, by Theorem 3.5. □

The following is a converse to Lemma 2.18.

**Lemma 4.13.** *Let $u, v \in \mathbb{Z}^2 \setminus 0$. If*

$$p \in \mathcal{C}^u \quad \text{and} \quad q \in \mathcal{C}^v$$

*are cycles for which $\bar{p} = \bar{q}$, then $u = v$.*

*Proof.* Suppose to the contrary that $u \neq v$. Then $p$ and $q$ intersect at some vertex $i$ since $u$ and $v$ are both nonzero. Let $p_i$ and $q_i$ be the respective cyclic permutations of $p$ and $q$ with tails at $i$. By assumption,

$$\bar{p}_i = \bar{p} = \bar{q} = \bar{q}_i.$$

Thus $p_i, q_i$ is a non-cancellative pair Lemma 4.12. Therefore $u = v$ by Lemma 2.4.1. □

Recall that if $p$ and $q$ are paths satisfying

$$\mathrm{t}(p^+) = \mathrm{t}(q^+) \quad \text{and} \quad \mathrm{h}(p^+) = \mathrm{h}(q^+),$$

then representatives $\tilde{p}^+$ and $\tilde{q}^+$ of their lifts bound a compact region $\mathcal{R}_{\tilde{p},\tilde{q}}$ in $\mathbb{R}^2$. If only one pair of representatives is considered, then by abuse of notation we denote $\mathcal{R}_{\tilde{p},\tilde{q}}$ by $\mathcal{R}_{p,q}$. Furthermore, we denote the interior of $\mathcal{R}_{p,q}$ by $\mathcal{R}^\circ_{p,q}$.

**Lemma 4.14.** *Let $p, q$ be a non-cancellative pair.*

*(1) Suppose $p$ and $q$ do not have proper subpaths (modulo $I$) which form a non-cancellative pair. If $r$ is a path of minimal length such that $rp = rq \neq 0$ (resp. $pr = qr \neq 0$), then each rightmost (resp. leftmost) arrow subpath of $r^+$ lies in the region $\mathcal{R}_{p,q}$ (modulo $I$).*

*(2) If the non-cancellative pair $p, q$ in Claim (2) is minimal, then $r^+$ lies in the region $\mathcal{R}_{p,q}$, and its head $\mathrm{h}(r^+)$ (resp. tail $\mathrm{t}(r^+)$) lies in the interior $\mathcal{R}^\circ_{p,q}$.*

*Proof.* (1) Suppose the hypotheses hold, with $r$ a path of minimal length such that $rp = rq \neq 0$. Assume to the contrary that a rightmost arrow subpath of $r^+$ does not lie in the interior of $\mathcal{R}_{p,q}$; see Figure 17.

By assumption, $p$ and $q$ do not have proper subpaths which form a non-cancellative pair. Thus there is a rightmost non-vertex subpath $p_2$ of $p$ (or $q$) and a leftmost non-vertex subpath $r_1$ of $r$ (modulo $I$) such that for some arrow $s$, $r_1 p_2 s$ is a unit cycle. Let $ts$ be the complementary unit cycle containing $s$.

Since $rp$ homotopes to $rq$, the path $t^+$ intersects the interior of $\mathcal{R}_{p,q}$. We thus have the setup given in Figure 17.ii. But then $q^+$ intersects the interiors of the unit cycles $(r_1 p_2 s)^+$ and $(ts)^+$ since $\mathrm{h}(q^+) = \mathrm{h}(p_2^+)$, a contradiction. Therefore $r_1^+$ lies in the interior of $\mathcal{R}_{p,q}$.

(2) Suppose $r^+$ intersects $p^+$; see Figure 18. Here $p$ and $r$ factor into (possibly vertex) paths $p = p_2 p_1$ and $r = r_2 r_1$, with $\mathrm{h}(p_1) = \mathrm{h}(r_1)$. We claim that the non-cancellative pair $p, q$ is not minimal.

Indeed, each rightmost arrow subpath of $r_1^+$ lies in $\mathcal{R}_{p,q}$ by Claim (1). Thus the path $r_1$ is not a vertex. Whence $r_1 p_2 \in \mathcal{C}^0$ is not a vertex. Therefore there is some $m \geq 1$ such that $\overline{r_1 p_2} = \sigma^m$ by Lemma 4.11.1. In particular, the paths $p_1 \sigma_i^m$ and $r_1 q$ satisfy

$$\bar{p}_1 \sigma^m = \overline{r_1 q}.$$

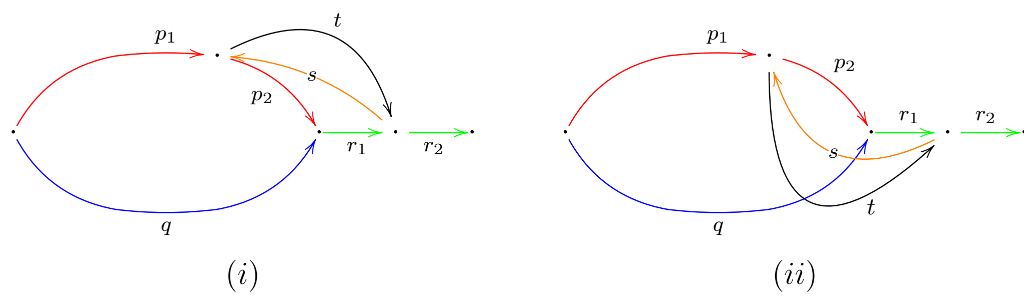


FIGURE 17. Setup for Lemma 4.14.1. In both cases, the paths $p = p_2p_1$, $q$, and $r = r_2r_1$ are drawn in red, blue, and green respectively. The orange path $s$ is an arrow, and the cycles $st$ and $sr_1p_2$ are unit cycles. This leads to a contradiction in case (ii) since $q$ passes through the interior of these unit cycles.

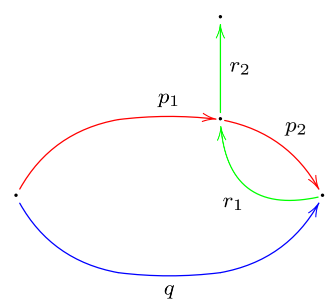


FIGURE 18. Setup for Lemma 4.14.2. The paths $p = p_3p_2p_1$, $q$, and $r_1$ are drawn in red, blue, and green respectively. This shows that the non-cancellative pair $p, q$ is not minimal.

Thus $p_1\sigma^m_{\mathrm{t}(p)}$ and $r_1q$ form a non-cancellative pair by Lemma 4.12. But by choosing a representative of the unit cycle $\sigma^+_{\mathrm{t}(p)}$ which lies in $\mathcal{R}_{p,q}$, it follows that

$$\mathcal{R}_{p_1\sigma^m_{\mathrm{t}(p)},r_1q}$$

is properly contained in $\mathcal{R}_{p,q}$. Therefore the non-cancellative pair $p, q$ is not minimal. $\square$

**Remark 4.15.** The assumption in Lemma 4.14.1 that $p$ and $q$ do not have proper subpaths $p'$ and $q'$ such that $\psi(p') = \psi(q')$ is necessary. Indeed, consider the subquiver given in Figure 19. Here, $\psi$ contracts the two arrows whose tails have indegree 1. Observe that $rp = rq \neq 0$, and $r^+$ does not intersect the interior of $\mathcal{R}_{p,q}$. However, $p$ and $q$ have proper subpaths $p'$ and $q'$ modulo $I$ satisfying $\psi(p') = \psi(q')$.

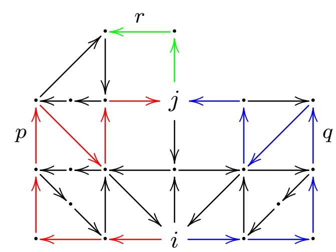


Figure 19. Setup for Remark 4.15. The paths $p$, $q$, $r$ are drawn in red, blue, and green respectively.

The following generalizes Lemma 2.19.1 and Proposition 2.20.2 to the case where $A$ is non-cancellative.

**Lemma 4.16.** *Let $u \in \mathbb{Z}^2 \setminus 0$. Suppose $a \in Q_1$, $p \in \hat{\mathcal{C}}^u_{\mathrm{t}(a)}$, and $q \in \hat{\mathcal{C}}^u_{\mathrm{h}(a)}$. If $\mathcal{R}^\circ_{ap,qa}$ contains no vertices, then $ap = qa$. Consequently, $\bar{p} = \bar{q}$.*

*Proof.* Suppose the hypotheses hold. If $(ap)^+$ and $(qa)^+$ have no cyclic subpaths (modulo $I$), then $ap = qa$ by Lemma 2.12.2.

So suppose $(qa)^+$ contains a cyclic subpath. The path $q^+$ has no cyclic subpaths since $q$ is in $\hat{\mathcal{C}}$. Thus $q$ factors into paths $q = q_2 q_1$, where $(q_1 a)^+$ is a cycle. In particular,

$$\mathrm{t}(p^+) = \mathrm{t}((q_1 q_2)^+) \quad \text{and} \quad \mathrm{h}(p^+) = \mathrm{h}((q_1 q_2)^+).$$

Whence $p$ and $q_1 q_2$ bound a compact region $\mathcal{R}_{p,q_1 q_2}$. Furthermore, its interior $\mathcal{R}^\circ_{p,q_1 q_2}$ contains no vertices since $\mathcal{R}^\circ_{ap,qa}$ contains no vertices.

The path $(q^2)^+$ has no cyclic subpaths, again since $q$ is in $\hat{\mathcal{C}}$. Thus $(q_1 q_2)^+$ also has no cyclic subpaths. Furthermore, $p^+$ has no cyclic subpaths since $p$ is in $\hat{\mathcal{C}}$. Therefore $p = q_1 q_2$ by Lemma 2.12.2.

Since there are no vertices in $\mathcal{R}^\circ_{ap,qa}$, there are also no vertices in the interior of the region bounded by the cycle $(aq_1)^+$. Thus there is some $\ell \geq 1$ such that

$$aq_1 = \sigma^\ell_{\mathrm{h}(a)} \quad \text{and} \quad q_1 a = \sigma^\ell_{\mathrm{t}(a)}.$$

Therefore

$$ap = aq_1 q_2 = \sigma^\ell_{\mathrm{h}(a)} q_2 \overset{(\mathrm{I})}{=} q_2 \sigma^\ell_{\mathrm{t}(a)} = q_2 q_1 a = qa,$$

where (I) holds by Lemma 1.5.

It thus follows that $\bar{p} = \bar{q}$ by the proof of Proposition 2.20.2 with $r = a$. $\square$

4.2. **Reduced and homotopy centers.** Throughout, $A$ is a non-cancellative dimer algebra and $\psi : A \to A'$ is a cyclic contraction. We denote by $Z$ and $Z'$ the respective centers of $A$ and $A'$.

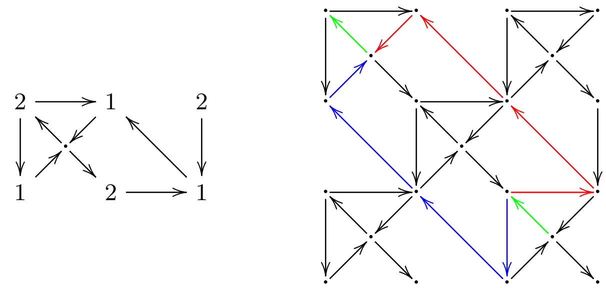


FIGURE 20. A dimer algebra $A$ for which $\operatorname{nil} Z \neq 0$. A fundamental domain of $Q$ is shown on the left, and a larger region of $Q^+$ is shown on the right. The paths $p$, $q$, $a$ are drawn in red, blue, and green respectively. The element $(p-q)a + a(p-q)$ is central and squares to zero.

4.2.1. *The central nilradical.* Recall that cancellative dimer algebras are prime (Proposition 3.11), and in particular have reduced centers (Theorem 3.3.4). In the following, we show that these properties do not necessarily hold in the non-cancellative case.

**Theorem 4.17.** *Dimer algebras exist with non-vanishing central nilradical. Consequently, dimer algebras exist which are not prime.*

*Proof.* Consider the non-cancellative dimer algebra $A$ with quiver $Q$ given in Figure 20. (A cyclic contraction of $A$ is given in Figure 1.) The paths $p$, $q$, $a$ satisfy

$$z := (p-q)a + a(p-q) \in \operatorname{nil} Z.$$

In particular, $\operatorname{nil} Z \neq 0$. $A$ is therefore not prime since

$$zAz = z^2 A = 0.$$

($A$ also contains non-central elements $a, b$ with the property that $aAb = 0$; for example, $(p-q)Ae_1 = 0$.) $\square$

**Question 4.18.** Is there a necessary and/or sufficient combinatorial condition for the central nilradical of a non-cancellative dimer algebra to vanish?

In the following, we characterize the central nilradical in terms of the cyclic contraction $\psi$, and show that it is a prime ideal of $Z$.

**Lemma 4.19.** *Suppose $p^+$ is a cycle in $Q^+$. If $p$ is not equal to a product of unit cycles (modulo $I$), then $p \notin Ze_{\mathrm{t}(p)}$.*

*Proof.* Suppose $p \in \mathcal{C}_i^0$ satisfies $p \neq \sigma_i^n$ for all $n \geq 1$. Two examples of such a cycle are given by the red cycles in Figures 21.i and 21.ii.

Let $r$ be a path with tail at $i$ whose lift $r^+$ does not intersect the interior of the region bounded by $p^+$ in $\mathbb{R}^2$, modulo $I$. Suppose $q$ is a cycle satisfying $rp = qr$. Then

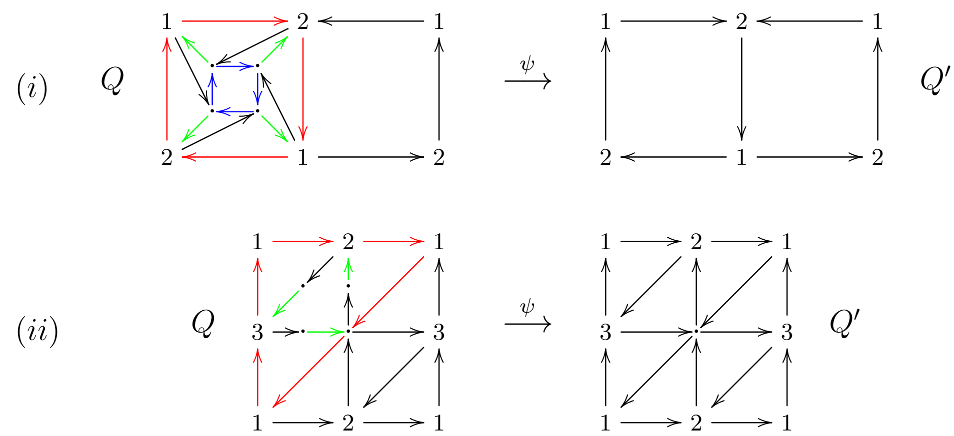


FIGURE 21. Examples for Remarks 4.20, 4.22, and Proposition 4.72. The quivers are drawn on a torus, the contracted arrows are drawn in green, and the 2-cycles have been removed from $Q'$. In each example, the cycle in $Q$ formed from the red arrows is not equal to a product of unit cycles (modulo $I$). However, in example (i) this cycle is mapped to a unit cycle in $Q'$ under $\psi$.

the region bounded by $q^+$ in $\mathbb{R}^2$ contains $p^+$. Furthermore, $rp = qr$ implies $\bar{p} = \bar{q}$ since $B$ is an integral domain.

But for a sufficiently long path $r^+$, the $\bar{\tau}\psi$-image of any cycle at $\mathrm{h}(r)$ whose lift contains $p^+$ in its interior will clearly not equal $\bar{p}$. Thus for such a path $r$, $rp \neq qr$ for all cycles $q$. Therefore $p$ is not in $Ze_i$. $\square$

**Remark 4.20.** It is possible for two cycles in $Q^+$, one of which is properly contained in the region bounded by the other, to have equal $\bar{\tau}\psi$-images. Indeed, consider Figure 21.i: the red cycle and the unit cycle in its interior both have $\bar{\tau}\psi$-image $\sigma$.

**Lemma 4.21.** *Let $u \in \mathbb{Z}^2 \setminus 0$. Suppose $p, q \in \mathcal{C}_i^u$ are cycles such that*

$$pq = qp \quad \text{and} \quad \psi(p) = \psi(q).$$

*Then*

$$p^2 = qp = q^2.$$

*Consequently*

$$(p-q)^2 = 0.$$

*Proof.* For brevity, if $sa$ and $ta$ are unit cycles with $a \in Q_1$, then we refer to $s$ as an arc and $t$ as its complementary arc.

Suppose the hypotheses hold. If $p = q$, then the lemma trivially holds, so suppose $p \neq q$. Since $pq = qp$, there are subpaths $q', q'', a, c$ of $q$ and a path $b$ such that

$$pq = p(cq') = (pc)q' = (q''b)q' = q''(bq') = q''(ap) = (q''a)p = qp.$$

In particular,

$$q = cq' = q''a, \quad ap = bq', \quad pc = q''b.$$

See Figure 22, where $p$ is drawn in red, $q$ is drawn in blue, and their common subpaths are drawn in violet. In case (i) $a$ is a rightmost subpath of $q$ alone, and in case (ii) $a$ is a rightmost subpath of both $p$ and $q$.

(i) First suppose $a$ is a rightmost subpath of $q$ alone.

We claim that $p^+$ and $q^+$ have no cycle subpaths. Indeed, if $p^+$ has a unit cycle subpath $\sigma_i^+$, then $\sigma_i^+$ can be homotoped to a unit cycle at $\mathrm{h}(p)$ which contains $a$ by Lemma 1.5, yielding case (ii). Otherwise, if $p^+$ contains a cyclic subpath $r^+$ which is not equal to a product of unit cycles, then $pq \neq qp$ by Lemma 4.19. This proves our claim.

Now suppose to the contrary that $ac \neq b$. Then, since $p \neq q$ and $pc = q''b$, there is an arc subpath of $pc$ (drawn in green in Figure 22) containing a leftmost arrow subpath of $c$ and rightmost arrow subpath of $p$, whose complementary arc (also drawn in green) lifts to a path that lies in the region bounded by $(pc)^+$ and $(q''b)^+$. In particular, the two cycles formed from the green paths and the orange arrow are unit cycles. But this is not possible since $\mathrm{t}(a) = \mathrm{t}(p)$, a contradiction. Thus $ac = b$, yielding

$$qp = q''ap = q''bq' = q''acq' = q^2. \tag{31}$$

Similarly $pq = p^2$.

(ii) Now suppose $a$ is a rightmost subpath of both $p$ and $q$. Then the relation $pc = q''b$ implies $ac = b$ by the contrapositive of Lemma 4.14.1. Therefore (31) holds. □

**Remark 4.22.** The assumption in Lemma 4.21 that $p^+$ and $q^+$ are not cycles is necessary. Indeed, if $p, q \in \mathcal{C}^0$, then it is possible for $pq = qp$, $\psi(p) = \psi(q)$, and $(p-q)^2 \neq 0$. For example, consider Figure 21.ii: the cycle $p \in e_3Ae_3$ formed from the red arrows satisfies

$$p\sigma_3^2 = \sigma_3^2 p \quad \text{and} \quad \psi(p) = \psi(\sigma_3^2).$$

However, $(p - \sigma_3^2)^m \neq 0$ for each $m \geq 1$.

**Lemma 4.23.** *Consider a central element*

$$z = \sum_{i \in Q_0} (p_i - q_i),$$

*where $p_i$, $q_i$ are elements in $e_iAe_i$. Then for each $i \in Q_0$,*

$$p_i q_i = q_i p_i.$$

*Proof.* For each $i \in Q_0$ we have

$$p_i^2 - p_i q_i = p_i(p_i - q_i) = p_i z = z p_i = (p_i - q_i)p_i = p_i^2 - q_i p_i.$$

Whence $p_i q_i = q_i p_i$. □

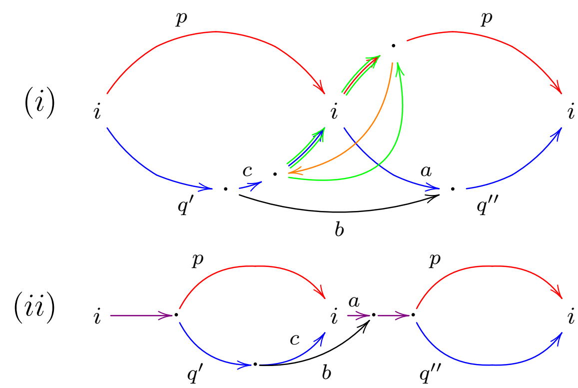


FIGURE 22. Cases for Lemma 4.21.

**Theorem 4.24.** *Let $A$ be a non-cancellative dimer algebra and $\psi : A \to A'$ a cyclic contraction. Then the central elements in the kernel of $\psi$ are precisely the nilpotent central elements of $A$,*

$$Z \cap \ker \psi = \operatorname{nil} Z.$$

*Proof.* (i) We first claim that if $z \in Z \cap \ker \psi$, then $z^2 = 0$, and in particular $z \in \operatorname{nil} Z$.

Consider a central element $z$ in $\ker \psi$. Since $z$ is central it commutes with the vertex idempotents, and so $z$ is a $k$-linear combination of cycles. Therefore, since $\psi$ sends paths to paths and $I'$ is generated by differences of paths, it suffices to suppose $z$ is of the form

$$z = \sum_{i \in Q_0} (p_i - q_i),$$

where $p_i$, $q_i$ are cycles in $e_i A e_i$ with equal $\psi$-images modulo $I'$. Note that there may be vertices $i \in Q_0$ for which $p_i = q_i = 0$.

By Lemma 4.23, for each $i \in Q_0$ we have

$$p_i q_i = q_i p_i.$$

Furthermore, by Lemmas 1.5 and 4.19 it suffices to suppose that the lifts $p_i^+, q_i^+$ are not cycles in $Q^+$. Therefore by Lemma 4.21,

$$z^2 = (\sum_{i \in Q_0} (p_i - q_i))^2 = \sum_{i \in Q_0} (p_i - q_i)^2 = 0.$$

Whence $z \in \operatorname{nil} Z$.

(ii) We now claim that if $z \in \operatorname{nil} Z$, then $z \in \ker \psi$.

Suppose $z^n = 0$. Then for each $i \in Q_0$,

$$\bar{\tau}\psi(ze_i)^n \overset{\text{(I)}}{=} \bar{\tau}\psi\left((ze_i)^n\right) \overset{\text{(II)}}{=} \bar{\tau}\psi\left(z^n e_i\right) = 0,$$

where (I) holds since $\bar{\tau}\psi$ is an algebra homomorphism on $e_iAe_i$, and (II) holds since $z$ is central. But $\bar{\tau}\psi\left(e_iAe_i\right)$ is contained in the integral domain $B$. Whence

$$\bar{\tau}\psi(ze_i) = 0.$$

Therefore $\psi(ze_i) = 0$ since $\bar{\tau}$ is injective, by Theorem 3.5. But then

$$\psi(z) \overset{\text{(I)}}{=} \psi\left(z \sum_{i \in Q_0} e_i\right) \overset{\text{(II)}}{=} \sum_{i \in Q_0} \psi(ze_i) = 0,$$

where (I) holds since the vertex idempotents form a complete set, and (II) holds since $\psi$ is a $k$-linear map. Therefore $\psi(z) = 0$. $\square$

4.2.2. *The reduced center as a subalgebra.* Let $\psi : A \to A'$ be a cyclic contraction. Consider the map

$$\tilde{\tau} : A \to M_{|Q_0|}(B)$$

defined on $p \in e_jAe_i$ by

$$\tilde{\tau}(p) = \bar{p}E_{ji} = \bar{\tau}\psi(p)E_{ji}, \tag{32}$$

and extended $k$-linearly to $A$.

**Lemma 4.25.** *The map $\tilde{\tau} : A \to M_{|Q_0|}(B)$ is an algebra homomorphism.*

*Proof.* By Lemma 1.6, $\tau : A' \to M_{|Q_0'|}(B)$ is an algebra homomorphism. Furthermore, $\psi$ is a $k$-linear map, and an algebra homomorphism when restricted to each vertex corner ring $e_iAe_i$. $\square$

In addition to the cycle algebra $S := k\left[\cup_{i \in Q_0} \bar{\tau}\psi(e_iAe_i)\right]$, which is generated by the union of the $\bar{\tau}\psi$-images of the vertex corner rings of $A$, it will also be useful to consider the algebra generated by their intersection:

**Definition 4.26.** The *homotopy center* of $A$ is the algebra

$$R := k\left[\cap_{i \in Q_0} \bar{\tau}\psi(e_iAe_i)\right] = \bigcap_{i \in Q_0} \bar{\tau}\psi(e_iAe_i).$$

**Theorem 4.27.** *Let $A$ be a non-cancellative dimer algebra and $\psi : A \to A'$ a cyclic contraction. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{nil} Z \hookrightarrow Z \xrightarrow{\bar{\psi}} R, \tag{33}$$

*where $\bar{\psi}$ is an algebra homomorphism. Therefore the reduction $\hat{Z} := Z/\operatorname{nil} Z$ of $Z$ is isomorphic to a subalgebra of $R$.*

*Proof.* (i) We first claim that the composition

$$\tau\psi : A \to A' \to M_{|Q_0'|}(B)$$

induces an algebra homomorphism

$$\bar{\psi} : Z \to R, \quad z \mapsto \overline{ze_i}, \tag{34}$$

where $i \in Q_0$ is any vertex.

Consider a central element $z \in Z$ and vertices $j, k \in Q_0$. By the construction of $Q$, there is a path $p$ from $j$ to $k$. For $i \in Q_0$, set $z_i := ze_i \in e_iAe_i$. By Lemma 4.25, $\bar{\tau}\psi$ is an algebra homomorphism on each vertex corner ring $e_iAe_i$. Thus

$$\bar{p}\bar{z}_j = \overline{pz_j} = \overline{pz} = \overline{zp} = \overline{z_kp} = \bar{z}_k\bar{p} \in B.$$

But the image $\bar{p}$ is nonzero since $\tau$ is an impression of $A'$ and the $\psi$-image of any path is nonzero. Thus, since $B$ is an integral domain,

$$\bar{z}_j = \bar{z}_k. \tag{35}$$

Therefore, since $j, k \in Q_0$ were arbitrary,

$$\bar{z}_j \in k\left[\cap_{i \in Q_0} \bar{\tau}\psi(e_iAe_i)\right] = R.$$

(ii.a) Let $z \in Z$ and $i \in Q_0$. We claim that $\psi(ze_i) = 0$ implies $\psi(z) = 0$. For each $j \in Q'_0$, denote by

$$c_j := \left|\psi^{-1}(j) \cap Q_0\right|$$

the number of vertices in $\psi^{-1}(j)$. Since $\psi$ maps $Q_0$ surjectively onto $Q'_0$, we have $c_j \geq 1$. Furthermore, if $k \in \psi^{-1}(j)$, then

$$\psi(z)e_j = c_j\psi(ze_k). \tag{36}$$

Set

$$z'_j := c_j^{-1}\psi(z)e_j.$$

Then the sum

$$z' := \sum_{j \in Q'_0} z'_j$$

is in the center $Z'$ of $A'$ by (35) and (28) in Theorem 3.5.[7] Therefore

$$\bar{\tau}(z'_j) = \bar{\tau}(z'e_j) = \bar{\tau}(z'e_{\psi(i)}) = \bar{\tau}(z'_{\psi(i)}) = \bar{\tau}(c^{-1}_{\psi(i)}\psi(z)e_{\psi(i)}) \overset{(\mathrm{I})}{=} \bar{\tau}(\psi(ze_i)) = 0,$$

where (I) holds by (36). Thus $z'_j = 0$ since $\bar{\tau}$ is injective. Whence

$$\psi(z)e_j = c_jz'_j = 0.$$

But this holds for each $j \in Q'_0$. Therefore $\psi(z) = 0$, proving our claim.

(ii.b) We now claim that the homomorphism (34) can be extended to the exact sequence (33). Indeed, $\bar{\psi} : Z \to R$ factors into the homomorphisms

$$Z \xrightarrow{\cdot e_i} Ze_i \xrightarrow{\psi} \psi(Ze_i) \xrightarrow{\bar{\tau}} R.$$

Suppose $z \in Z$ is in the kernel of $\bar{\psi}$,

$$\bar{\tau}\psi(ze_i) = \bar{\psi}(z) = 0.$$

[7] Note that $\psi(z)$ is not in $Z'$ if there are vertices $i, j \in Q'_0$ for which $c_i \neq c_j$. Therefore in general $\psi(Z)$ is not contained in $Z'$.

By Theorem 3.5, $\bar{\tau}$ is injective on each vertex corner ring $e_iAe_i$. Thus $\psi(ze_i) = 0$. Whence $\psi(z) = 0$ by Claim (ii.a). Therefore $z \in \operatorname{nil} Z$ by Theorem 4.24. This proves our claim. $\square$

**Corollary 4.28.** *The algebras $\hat{Z}$, $R$, and $S$ are integral domains. Therefore the central nilradical* $\operatorname{nil} Z$ *of $A$ is a prime ideal of $Z$, and the schemes*

$$\operatorname{Spec} Z \quad \textit{and} \quad \operatorname{Spec} \hat{Z}$$

*are irreducible.*

*Proof.* $R$ and $S$ are domains since they are subalgebras of the domain $B$. Therefore $\hat{Z}$ is a domain since it isomorphic to a subalgebra of $R$ by Theorem 4.27. $\square$

**Lemma 4.29.** *If $g$ is a monomial in $B$ and $g\sigma$ is in $S$, then $g$ is also in $S$.*

*Proof.* Suppose the hypotheses hold. Then there is a cycle $p \in A'$ such that $\bar{p} = g\sigma$. Let $u \in \mathbb{Z}^2$ be such that $p \in \mathcal{C}'^u$. Since $A'$ is cancelative, there is a cycle $q \in \hat{\mathcal{C}}'^u$ by Proposition 2.10. Furthermore, $\sigma \nmid \bar{q}$ by Proposition 3.10. Thus there is some $m \geq 1$ such that

$$\bar{q}\sigma^m = \bar{p} = g\sigma,$$

by Lemma 2.18. Therefore

$$g = (g\sigma)\sigma^{-1} = \bar{q}\sigma^{m-1} \in S' \overset{(\text{I})}{=} S,$$

where (I) holds since $\psi$ is cyclic. $\square$

**Proposition 4.30.**

(1) *If $g \in R$ and $\sigma \nmid g$, then $g \in \bar{\psi}(Z)$.*
(2) *If $g \in S$, then there is some $N \geq 0$ such that for each $n \geq 1$, $g^n\sigma^N \in \bar{\psi}(Z)$.*
(3) *If $g \in R$, then there is some $N \geq 1$ such that $g^N \in \bar{\psi}(Z)$.*

*Proof.* Since $R$ is generated by monomials, it suffices to consider a non-constant monomial $g \in R$. Then for each $i \in Q_0$, there is a cycle $c_i \in e_iAe_i$ satisfying $\bar{c}_i = g$.

(1) Suppose $\sigma \nmid g$. Fix $a \in Q_1$, and set

$$p := c_{\mathrm{t}(a)} \quad \text{and} \quad q := c_{\mathrm{h}(a)}.$$

See Figure 23. We claim that $ap = qa$.

Let $u, v \in \mathbb{Z}^2$ be such that

$$p \in \mathcal{C}^u \quad \text{and} \quad q \in \mathcal{C}^v.$$

By assumption, $\sigma \nmid g = \bar{p} = \bar{q}$. Then $u$ and $v$ are both nonzero by Lemma 4.11.1. Thus $u = v$ by Lemma 4.13. Therefore $(ap)^+$ and $(qa)^+$ bound a compact region $\mathcal{R}_{ap,qa}$ in $\mathbb{R}^2$.

We proceed by induction on the number of vertices in the interior $\mathcal{R}^\circ_{ap,qa}$.

First suppose there are no vertices in $\mathcal{R}^\circ_{ap,qa}$. Since $\sigma \nmid g = \bar{p} = \bar{q}$, $p$ and $q$ are in $\hat{\mathcal{C}}$ by Lemma 4.11.2. Therefore $ap = qa$ by Lemma 4.16.

So suppose $\mathcal{R}^{\circ}_{ap,qa}$ contains at least one vertex $i^+$. Let $w \in \mathbb{Z}^2$ be such that $c_i \in \mathcal{C}^w$. Then $w = u = v$, again by Lemma 4.13. Therefore $c_i$ intersects $p$ at least twice or $q$ at least twice. Suppose $c_i$ intersects $p$ at vertices $j$ and $k$. Then $p$ factors into paths

$$p = p_2 e_k t e_j p_1 = p_2 t p_1.$$

Let $s^+$ be the subpath of $(c_i^2)^+$ from $j^+$ to $k^+$. Then

$$\mathrm{t}(s^+) = \mathrm{t}(t^+) \quad \text{and} \quad \mathrm{h}(s^+) = \mathrm{h}(t^+).$$

In particular, $s^+$ and $t^+$ bound a compact region $\mathcal{R}_{s,t}$.

Since we are free to choose the vertex $i^+$ in $\mathcal{R}^{\circ}_{ap,qa}$, we may suppose $\mathcal{R}^{\circ}_{s,t}$ contains no vertices. Furthermore, $c_i^+$ and $p^+$ have no cyclic subpaths since $\sigma \nmid g$, again by Lemma 4.11.2. Thus their respective subpaths $s^+$ and $t^+$ have no cyclic subpaths. Whence $s = t$ by Lemma 2.12.2.

Furthermore, since $\mathcal{R}^{\circ}_{ap_2sp_1,qa}$ contains less vertices than $\mathcal{R}^{\circ}_{ap,qa}$, it follows by induction that

$$ap_2sp_1 = qa.$$

Therefore

$$ap = a(p_2tp_1) = a(p_2sp_1) = qa.$$

Since $a \in Q_1$ was arbitrary, the sum $\sum_{i \in Q_0} c_i$ is central in $A$.

(2) Fix an arrow $a \in Q_1$. Set $i := \mathrm{t}(a)$ and $j := \mathrm{h}(a)$. Let $r^+$ be a path in $Q^+$ from $\mathrm{h}\left((ac_i)^+\right)$ to $\mathrm{t}\left((ac_i)^+\right)$. Then by Lemma 2.3.1, there is some $\ell, m, n \geq 0$ such that

$$\sigma_i^m = rc_ja\sigma_i^\ell \quad \text{and} \quad ac_ir\sigma_j^\ell = \sigma_j^n.$$

Thus

$$\sigma^m = \bar{\tau}\psi\left(rc_ja\sigma_i^\ell\right) = \bar{r}\bar{c}_j\bar{a}\sigma^\ell = \bar{a}\bar{c}_i\bar{r}\sigma^\ell = \bar{\tau}\psi\left(ac_ir\sigma_j^\ell\right) = \sigma^n.$$

Furthermore, $\sigma \neq 1$ since $\bar{\tau}$ is injective. Whence $n = m$ since $B$ is an integral domain. Therefore

$$a\left(c_i\sigma_i^n\right) = ac_i\left(rc_ja\sigma_i^\ell\right) \overset{\text{(I)}}{=} \left(ac_ir\sigma_j^\ell\right)c_ja = \sigma_j^nc_ja \overset{\text{(II)}}{=} \left(c_j\sigma_i^n\right)a, \tag{37}$$

where (I) and (II) hold by Lemma 1.5.

For each $a \in Q_1$ there is an $n = n(a)$ such that (37) holds. Set

$$N := \max\left\{n(a) \mid a \in Q_1\right\}.$$

Then (37) implies that the element $\sum_{i \in Q_0} c_i\sigma_i^N$ is central.

Now fix $n \geq 2$ and an arrow $a \in Q_1$. Again set $i := \mathrm{t}(a)$ and $j := \mathrm{h}(a)$. Then

$$ac_i^n\sigma_i^N = ac_i^{n-1}\left(c_i\sigma_i^N\right) = \left(c_j\sigma_j^N\right)ac_i^{n-1} \overset{\text{(I)}}{=} c_jac_i^{n-1}\sigma_i^N = \cdots = c_j^n\sigma_h^Na,$$

where (I) holds by Lemma 1.5. Therefore, for each $n \geq 1$, the element

$$\sum_{i \in Q_0} c_i^n\sigma_i^N$$

is central. But its $\bar{\tau}\psi$-image is $g^n\sigma^N$, proving Claim (2).

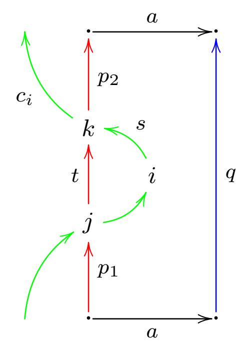


FIGURE 23. Setup for Proposition 4.30.1. The cycles $p = p_2 t p_1$, $q$, $c_i$, are drawn in red, blue, and green respectively. The path $a$ is an arrow.

(3) By Claim (1), it suffices to suppose $\sigma \mid g$. Then there is a monomial $h \in B$ such that

$$g = h\sigma.$$

By Lemma 4.29, $h$ is in $S$. Therefore by Claim (2), there is some $N \geq 1$ such that

$$g^N = h^N \sigma^N \in \bar{\psi}(Z).$$

□

The following theorem shows that it is possible for the reduced center $\hat{Z}$ to be properly contained in the homotopy center $R$. However, we will show that they determine the same nonlocal variety (Theorem 4.68), and that their integral closures are isomorphic (Theorem 4.75).

**Theorem 4.31.** *There exists dimer algebras for which the containment $\hat{Z} \hookrightarrow R$ is proper.*

*Proof.* Consider the contraction given in Figure 24. This contraction is cyclic:

$$S = k[x^2, xy, y^2, z] = S'.$$

We claim that the reduced center $\hat{Z}$ of $A = kQ/I$ is not isomorphic to $R$. By the exact sequence (33), it suffices to show that the homomorphism $\bar{\psi} : Z \hookrightarrow R$ is not surjective.

We claim that the monomial $z\sigma$ is in $R$, but is not in the image $\bar{\psi}(Z)$. It is clear that $z\sigma$ is in $R$ from the $\bar{\tau}\psi$ labeling of arrows given in Figure 24.

Assume to the contrary that $z\sigma \in \bar{\psi}(Z)$. Then by (35), for each $j \in Q_0$ there is an element $c_j$ in $Ze_j$ whose $\bar{\tau}\psi$-image is $z\sigma$. Consider the vertex $i \in Q_0$ shown in Figure

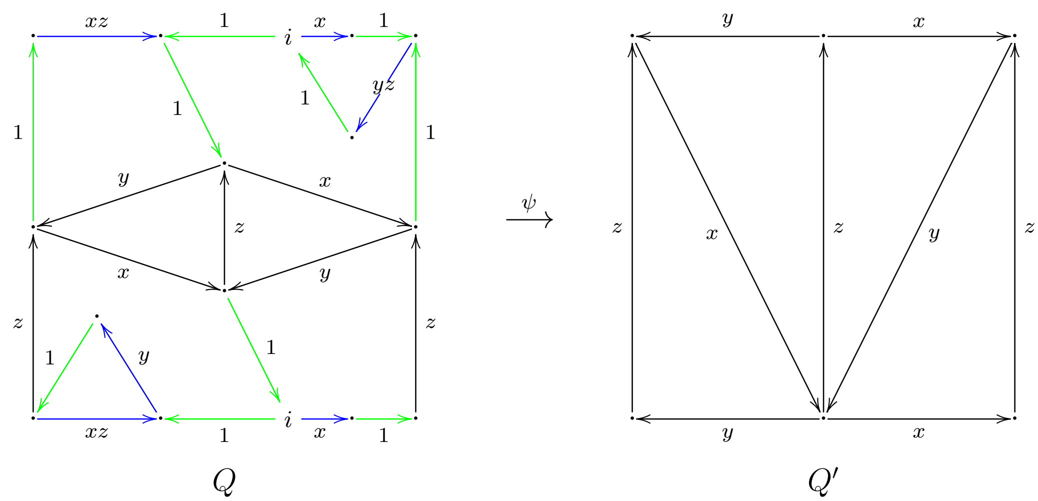


FIGURE 24. A cyclic contraction $\psi : A \to A'$ for which $\hat{Z} \subsetneq R$. $Q$ and $Q'$ are drawn on a torus, and the contracted arrows are drawn in green. The arrows drawn in blue form removable 2-cycles under $\psi$. The arrows in $Q$ are labeled by their $\bar{\tau}\psi$-images, and the arrows in $Q'$ are labeled by their $\bar{\tau}$-images.

24. The set of cycles in $e_iAe_i$ with $\bar{\tau}\psi$-image $z\sigma$ are as follows:

$$\begin{aligned}
p_1 &:= \delta_5 a_4 a_3 (\delta_1 a_3) \delta_6 b_1 \delta_3, \\
p_2 &:= \delta_2 b_4 (\delta_1 a_3) \delta_7 b_2, \\
p_3 &:= \delta_2 b_4 (\delta_1 a_3) \delta_1 a_2 \delta_4 \delta_3 \\
p_4 &:= \delta_5 a_6 a_3 (\delta_1 a_3) \delta_7 b_2, \\
p_5 &:= \delta_5 a_6 a_3 (\delta_1 a_3) \delta_1 a_2 \delta_4 \delta_3.
\end{aligned}$$

Thus for some coefficients $\alpha_1, \ldots, \alpha_5 \in k$,

$$c_i = \sum_{\ell=1}^{5} \alpha_\ell p_\ell.$$

Since $\bar{c}_i = z\sigma$, there is some $\ell$ for which $\alpha_\ell$ is nonzero. In particular, there are cycles $p'_\ell$ and $p''_\ell$ satisfying

$$b_2 p_\ell = p'_\ell b_2 \quad \text{and } \delta_3 p_\ell = p''_\ell \delta_3.$$

However, there are no cycles $p'_1, p''_2, p'_3, p''_4, p'_5$, for which

$$b_2 p_1 = p'_1 b_2, \quad \delta_3 p_2 = p''_2 \delta_3, \quad b_2 p_3 = p'_3 b_2, \quad \delta_3 p_4 = p''_4 \delta_3, \quad b_2 p_5 = p'_5 b_2.$$

Thus no such element $c_i \in Ze_i$ can exist, a contradiction. Therefore $\hat{Z} \ncong R$. $\square$

Theorem 4.31 raises the following question.

**Question 4.32.** Are there necessary and sufficient conditions for the isomorphism $\hat{Z} \cong R$ to hold?

4.3. **Homotopy dimer algebras.** We introduce the following class of algebras. Recall the definition of a non-cancellative pair (Definition 2.2).

**Definition 4.33.** Let $A$ be a dimer algebra. We call the quotient algebra

$$\tilde{A} := A / \langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle$$

the *homotopy (dimer) algebra* of $A$.

Note that a dimer algebra $A$ equals its homotopy algebra $\tilde{A}$ if and only if $A$ is cancellative.

Let $\psi : A \to A'$ be a cyclic contraction. Recall the algebra homomorphism

$$\tilde{\tau} : A \to M_{|Q_0|}(B)$$

defined in (32) (Lemma 4.25). This homomorphism induces an algebra monomorphism on the quotient $\tilde{A}$,

$$\tilde{\tau} : \tilde{A} \to M_{|Q_0|}(B),$$

by Lemma 4.12.

**Remark 4.34.** The ideal

$$\langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle \subset A$$

is contained in the kernel of $\psi$, but not conversely. Indeed, if $\psi$ contracts an arrow $\delta$, then $\delta - e_{\mathrm{t}(\delta)}$ is in the kernel of $\psi$, but $\delta$ and $e_{\mathrm{t}(\delta)}$ do not form a non-cancellative pair.

**Theorem 4.35.** *The algebra monomorphism $\tilde{\tau} : \tilde{A} \to M_{|Q_0|}(B)$ is an impression of the homotopy dimer algebra $\tilde{A}$. Consequently, the center of $\tilde{A}$ is isomorphic to the homotopy center $R$ of $A$.*

*Proof.* Denote by $\tilde{Z}$ the center of $\tilde{A}$.

(i) For generic $\mathfrak{b} \in \operatorname{Max} B$, the composition $\epsilon_{\mathfrak{b}}\tilde{\tau}$ (defined in (8)) is surjective by Claim (ii) in the proof of Theorem 3.5. Furthermore, the morphism

$$\operatorname{Max} B \to \operatorname{Max} \tau(\tilde{Z}), \quad \mathfrak{b} \mapsto \mathfrak{b}1_{|Q_0|} \cap \tau(\tilde{Z}),$$

is surjective by Claim (iii) in the proof of Theorem 3.5. Therefore $(\tilde{\tau}, B)$ is an impression of $\tilde{A}$.

(ii) Since $(\tilde{\tau}, B)$ is an impression of $\tilde{A}$, $\tilde{Z}$ is isomorphic to $R$ [B, Lemma 2.1 (2)]. $\square$

Recall that non-cancellative dimer algebras exist which are not prime, by Theorem 4.17. However, their homotopy algebras will always be prime:

**Proposition 4.36.** *Homotopy dimer algebras are prime.*

*Proof.* Follows similar to Proposition 3.11, since $\tilde{\tau} : \tilde{A} \to M_{|Q_0|}(B)$ is injective by Lemma 4.12. $\square$

4.4. **Non-annihilating paths.** Throughout, $\psi : A \to A'$ is a cyclic contraction. Recall the subset of arrows $Q_1^{\mathcal{S}} \subseteq Q_1$ defined in (25), and the definition of a minimal non-cancellative pair (Definition 2.2).

**Proposition 4.37.** *Let $p, q$ be a minimal non-cancellative pair. If $r$ is a path of minimal length satisfying*

$$rp = rq \neq 0 \quad \text{or} \quad pr = qr \neq 0,$$

*then each arrow subpath of $r$ is in $Q_1^{\mathcal{S}}$. In particular, if $A$ is non-cancellative, then $Q_1^{\mathcal{S}} \neq \emptyset$.*

*Proof.* Set $\sigma := \prod_{D \in \mathcal{P}} x_D$, and recall the algebra homomorphism $\eta$ defined in (3). Assume to the contrary that there is an arrow subpath $r_2$ of $r = r_3 r_2 r_1$ which is contained in a simple matching $D$; here $r_1$ and $r_3$ are paths, possibly vertices.

Since $D$ is a simple matching, there is a path $t'$ whose arrow subpaths are not contained in $D$, from $\mathrm{t}(p)$ to $\mathrm{h}(r_2)$. By Lemma 4.14.2, the path $r^+$ lies in $\mathcal{R}_{p,q}$, and the vertex $\mathrm{h}(r^+)$ lies in the interior $\mathcal{R}^{\circ}_{p,q}$. Thus the vertex $\mathrm{h}(r_2^+)$ also lies in $\mathcal{R}^{\circ}_{p,q}$. Therefore there is a leftmost non-vertex subpath $t^+$ of $t'^+$ contained in $\mathcal{R}_{p,q}$ with tail on $p$ or $q$; suppose $t$ has tail on $p$ and has minimal length. Then $p$ factors into paths $p = p_2 p_1$, where $\mathrm{h}(p_1^+) = \mathrm{t}(t^+)$, as shown in Figure 25.

Consider the path $s$ such that $(r_2 s)^+$ is a unit cycle that lies in the region $\mathcal{R}_{t, r_2 r_1 p_2}$. (Note that $s$ and $r_1 p$ may share arrows.) Then $x_D \nmid \bar{\eta}(s)$ since $r_2 \in D$. Whence

$$\sigma \nmid \bar{\eta}(s). \tag{38}$$

Furthermore, $x_D \nmid \bar{\eta}(t)$ since no arrow subpath of $t$ is contained in $D$. Thus

$$x_D \nmid \bar{\eta}(s)\bar{\eta}(t) = \bar{\eta}(st).$$

Therefore

$$\sigma \nmid \bar{\eta}(st). \tag{39}$$

Since $t^+$ lies in $\mathcal{R}_{p,q}$ and $\mathrm{h}(t^+)$ lies in $\mathcal{R}^{\circ}_{p,q}$, we have

$$\mathcal{R}_{r_1 p_2, st} \subsetneq \mathcal{R}_{p,q} \quad \text{and} \quad \mathcal{R}_{st p_1, r_1 q} \subsetneq \mathcal{R}_{p,q}.$$

Furthermore, by (39) and Lemma 2.3.2, there is some $m \geq 0$ such that

$$\sigma^m \bar{\eta}(st) = \bar{\eta}(r_1 p_2).$$

Thus, by Lemma 2.3.3 and the minimality of the pair $p, q$,

$$\sigma^m_{\mathrm{h}(s)} st = r_1 p_2.$$

Therefore

$$\sigma^m_{\mathrm{h}(s)} st p_1 = r_1 p. \tag{40}$$

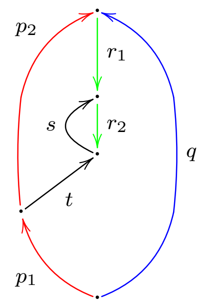


FIGURE 25. Setup for Proposition 4.37.

Since $p, q$ is a non-cancellative pair, $\bar{\eta}(p) = \bar{\eta}(q)$ by Lemma 2.3.3. Whence (40) implies

$$\sigma^m \bar{\eta}(stp_1) = \bar{\eta}(r_1 p) = \bar{\eta}(r_1)\bar{\eta}(p) = \bar{\eta}(r_1)\bar{\eta}(q) = \bar{\eta}(r_1 q).$$

Therefore, again by Lemma 2.3.3 and the minimality of the pair $p, q$,

$$\sigma^m_{\mathrm{h}(s)} stp_1 = r_1 q. \tag{41}$$

Consequently,

$$r_1 p \overset{(40)}{=} \sigma^m_{\mathrm{h}(s)} stp_1 \overset{(41)}{=} r_1 q.$$

But $r_2$ is an arrow. Therefore $r$ does not have minimal length such that $rp = rq$, contrary to our choice of $r$. $\square$

**Theorem 4.38.** *Suppose $\psi : A \to A'$ is a cyclic contraction. Then*

$$Q_1^* \subseteq Q_1^{\mathcal{S}}.$$

*Proof.* Suppose the hypotheses hold, and assume to the contrary that there is an arrow $\delta \in Q_1^* \setminus Q_1^{\mathcal{S}}$.

Let $s$ be a path for which $s\delta$ is a unit cycle. Since $\delta$ is contracted to a vertex, $\psi(s)$ is a unit cycle in $Q'$. Whence

$$\bar{\tau}\psi(s) = \sigma. \tag{42}$$

Since $D$ is simple, there is a cycle $p \in A$ supported on $Q \setminus D$ which contains $s$ as a subpath, as well as each arrow in $Q_1 \setminus D$. Since $s$ is a subpath of $p$, $\psi(s)$ is a subpath of $\psi(p)$. In particular,

$$\bar{\tau}\psi(s) \mid \bar{\tau}\psi(p) = \bar{p}.$$

Therefore (42) implies

$$\sigma \mid \bar{p}. \tag{43}$$

Since $A'$ is cancellative, $S'$ is generated by $\sigma$ and a set of monomials in $B$ not divisible by $\sigma$, by Proposition 3.10. Thus (43) implies that the monomial $\bar{p}/\sigma$ is in $S'$. Therefore, since $S = S'$, there is a cycle $q \in A$ such that

$$\bar{q} = \bar{p}/\sigma. \tag{44}$$

Set $i := \mathrm{t}(q)$. Since $p$ contains each arrow in $Q_1 \setminus D$ and $D$ is a perfect matching, $p$ also contains each vertex in $Q$. Thus we may assume that $\mathrm{t}(p) = i$, by possibly cyclically permuting the arrow subpaths of $p$. Furthermore, $\bar{\tau} : e_iAe_i \to B$ is injective by Theorem 3.5. Therefore (44) implies

$$\psi(p) = \psi(q\sigma_i). \tag{45}$$

Since $p$ contains each vertex in $Q$, $\bar{p}$ is in $R$. Thus there is some $n \geq 1$ such that $p^n \in Ze_i$ by Proposition 4.30.3. Whence

$$p^n(q\sigma_i)^n = (q\sigma_i)^n p^n.$$

Furthermore, $\psi(p^n) = \psi(q^n\sigma_i^n)$ by (45). Therefore by Lemma 4.21,

$$p^{2n} = q^{2n}\sigma_i^{2n}. \tag{46}$$

Now let $V$ be an $A$-module with support $Q \setminus D$. Then $p^{2n}$ does not annihilate $V$ since $p^{2n}$ is supported on $Q \setminus D$. However, $\sigma_i$ contains an arrow in $D$ since $D$ is a perfect matching. Thus $q^{2n}\sigma_i^{2n}$ annihilates $V$. But this is a contradiction to (46). □

**Lemma 4.39.** *If an arrow annihilates a simple $A$-module of dimension $1^{Q_0}$, then it is contained in a simple matching of $A$.*

*Proof.* Let $V_\rho$ be a simple $A$-module of dimension $1^{Q_0}$, and suppose $\rho(a) = 0$. Let $i \in Q_0$. Since $V_\rho$ is simple of dimension $1^{Q_0}$, there is a path $p$ from $\mathrm{t}(a)$ to $i$ such that $\rho(p) \neq 0$. Furthermore, $\sigma_i p = p\sigma_{\mathrm{t}(a)}$ by Lemma 1.5. Thus, since $a$ is a subpath $\sigma_{\mathrm{t}(a)}$ (modulo $I$), we have

$$\rho(\sigma_i)\rho(p) = \rho(\sigma_i p) = \rho(p\sigma_{\mathrm{t}(a)}) = 0.$$

Whence

$$\rho(\sigma_i) = 0.$$

Thus each unit cycle contains at least one arrow that annihilates $V_\rho$. Therefore there are perfect matchings $D_1, \ldots, D_m \in \mathcal{P}$ such that $V_\rho$ is supported on $Q \setminus (D_1 \cup \cdots \cup D_m)$.

Moreover, since $\rho(a) = 0$, there is some $1 \leq \ell \leq m$ such that $D_\ell$ contains $a$. Since $V_\rho$ is simple, there is a path $r$ supported on $Q \setminus (D_1 \cup \cdots \cup D_m)$ which passes through each vertex of $Q$. In particular, $r$ is supported on $Q \setminus D_\ell$. Therefore $D_\ell$ is a simple matching containing $a$. □

**Lemma 4.40.** *Let $\psi : A \to A'$ be a cyclic contraction. Then $R = S$ if and only if $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$.*

*Proof.* ($\Rightarrow$) Suppose $R = S$. Let $u \in \mathbb{Z}^2 \setminus 0$. Since $A'$ is cancellative, there is a cycle $q \in \hat{\mathcal{C}}'^u$, by Proposition 2.10. In particular, $\sigma \nmid \bar{q}$ by Proposition 2.20.1.

Since $R = S$, for each $i \in Q_0$ there is a cycle $p \in \mathcal{C}_i$ such that $\bar{p} = \bar{q}$. Furthermore, since $A$ admits a cyclic contraction, $A$ does not contain a non-cancellative pair where one of the paths is a vertex, by Lemma 4.8.3. Thus

$$p \in \hat{\mathcal{C}}_i$$

since $\sigma \nmid \bar{p}$, by Lemma 2.8.3. Moreover, $\psi(p) \in \mathcal{C}'^u$ since $\bar{\tau}(\psi(p)) = \bar{\tau}(q)$, by Lemma 2.18. Whence

$$p \in \mathcal{C}^u,$$

by Lemma 4.8.1. Therefore $p \in \hat{\mathcal{C}}_i^u$. In particular, $\hat{\mathcal{C}}_i^u \neq \emptyset$. The claim then follows since $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$ were arbitrary.

($\Leftarrow$) Follows from Proposition 2.20.2. □

**Theorem 4.41.** *Let $A = kQ/I$ be a dimer algebra. The following are equivalent:*

*(1) $A$ is cancellative.*
*(2) Each arrow annihilates a simple $A$-module of dimension $1^{Q_0}$.*
*(3) Each arrow is contained in a simple matching, $Q_1^{\mathcal{S}} = \emptyset$.*
*(4) If $\psi : A \to A'$ is a cyclic contraction, then $R = S$.*
*(5) If $\psi : A \to A'$ is a cyclic contraction, then $\psi$ is trivial, $Q_1^* = \emptyset$.*

*Proof.* (2) $\Leftrightarrow$ (3): Lemma 4.39.

(3) $\Rightarrow$ (1): Proposition 4.37.

(4) $\Rightarrow$ (3): Suppose $R = S$. Then $\hat{\mathcal{C}}_i^u \neq \emptyset$ for each $u \in \mathbb{Z}^2 \setminus 0$ and $i \in Q_0$, by Lemma 4.40. Therefore $Q_1^{\mathcal{S}} = \emptyset$ by Theorem 2.24.

(1) $\Rightarrow$ (4): If $A$ is cancellative, then $R = S$ by Theorem 3.5.

(1) $\Rightarrow$ (5): Suppose $A$ is cancellative, and $\psi : A \to A'$ is a cyclic contraction. Then

$$Q_1^* \overset{\text{(I)}}{\subseteq} Q_1^{\mathcal{S}} \overset{\text{(II)}}{=} \emptyset,$$

where (I) holds by Theorem 4.38, and (II) holds by the implication (1) $\Rightarrow$ (2). Therefore $Q_1^* = \emptyset$.

(5) $\Rightarrow$ (4): Clear. □

Non-cancellative dimer algebras do not necessarily admit contractions to cancellative dimer algebras, cyclic or not. In the following, we use Theorem 4.41 to show that if a dimer algebra has a permanent 2-cycle (Definition 2.5), then it cannot be contracted to a cancellative dimer algebra.

**Lemma 4.42.** *Suppose $\psi : A \to A'$ is a contraction of dimer algebras, and $A'$ has a perfect matching. Further suppose $ab$ is a 2-cycle in both $A$ and $A'$. Then $ab$ is removable in $A$ if and only if $ab$ is removable in $A'$.*

*Proof.* Suppose $ab$ is a permanent 2-cycle. Then $ab$ is given in Figure 3.ii or Figure 3.iii, by Lemma 2.6. In either case, $\psi(ab)$ is also a permanent 2-cycle by Lemma 4.8.1. □

**Proposition 4.43.** *Let $A$ be a dimer algebra with a permanent 2-cycle. Then $A$ is non-cancellative, and does not admit a contraction (cyclic or not) to a cancellative dimer algebra.*

*Proof.* (i) We first claim that $A$ is non-cancellative. By Lemma 2.6, there are two types of permanent 2-cycles, and these are given in Figures 3.ii and 3.iii. Let $D$ be a perfect matching of $A$. Then either $a$ or $b$ is contained in $D$. Therefore no arrow subpath of $p$ (resp. $p$ or $q$) in Figure 3.ii (resp. Figure 3.iii) is contained in $D$. In particular, no arrow subpath of $p$ (resp. $p$ or $q$) is contained in a simple matching. Whence $Q_1^{\mathcal{S}} \neq \emptyset$. Therefore $A$ is non-cancellative by Theorem 4.41.

(ii) By Claim (i) and Lemma 4.42, any contraction $\psi : A \to A'$ to a cancellative dimer algebra necessarily contracts the unit cycle $ab$ to a vertex. However, this is not possible by Lemma 4.7. □

4.5. **Nonnoetherian and nonlocal.** Throughout, $A$ is a dimer algebra, non-cancellative unless stated otherwise, and $\psi : A \to A'$ is a cyclic contraction. Let

$$\tilde{A} := A / \langle p - q \mid p, q \text{ is a non-cancellative pair} \rangle$$

and $R = Z(\tilde{A})$ be the homotopy algebra and homotopy center of $A$, respectively.

**Lemma 4.44.** *If $p$ is a cycle such that $\bar{p} \notin R$ and $\sigma \nmid \bar{p}$, then for each $n \geq 1$,*

$$\bar{p}^n \notin R.$$

*Furthermore, if $A$ is non-cancellative, then such a cycle exists.*

*Proof.* (i) Assume to the contrary that there is a cycle $p \in e_iAe_i$ such that $\bar{p} \notin R$, $\sigma \nmid \bar{p}$, and $\bar{p}^n \in R$ for some $n \geq 2$. Let $u \in \mathbb{Z}^2$ be such that $p \in \mathcal{C}_i^u$. Since $\bar{p}$ is not in $R$, there is a vertex $j \in Q_0$ such that

$$\bar{p} \notin \bar{\tau}\psi\left(e_jAe_j\right). \tag{47}$$

Furthermore, since $\bar{p}^n$ is in $R$, $p^n$ homotopes to a cycle $q \in e_iAe_i$ that passes through $j$,

$$q = p^n \quad (\text{modulo } I). \tag{48}$$

For $v \in \mathbb{Z}^2$, denote by $q_v^+ \in \pi^{-1}(q)$ the preimage with tail

$$\mathrm{t}(q_v^+) = \mathrm{t}(q^+) + v \in Q_0^+.$$

It is clear (see Figure 26) that there is a path $r^+$ from $j^+$ to $j^+ + u$, constructed from subpaths of $q^+$, $q_u^+$, and possibly $q_{mu}^+$ for some $m \in \mathbb{Z}$. In particular, the cycle $r := \pi(r^+) \in e_jAe_j$ is in $\mathcal{C}_j^u$.

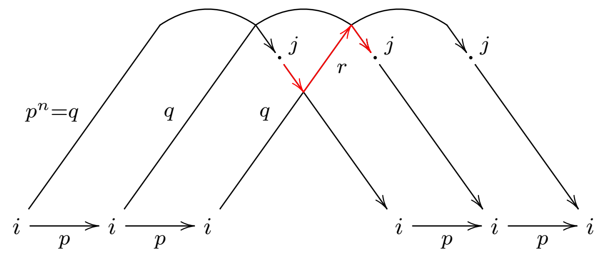


FIGURE 26. Setup for Lemma 4.44. The path $r^+$ is drawn in red. Its projection $r = \pi(r^+)$ to $Q$ is a cycle at $j$.

Furthermore, since $\sigma \nmid \bar{p}$, there is a simple matching $D$ such that $x_D \nmid \bar{p}$. Whence $x_D \nmid \bar{q}$ by (48). Thus $x_D \nmid \bar{r}$, and so $\sigma \nmid \bar{r}$. In particular,

$$\sigma \nmid \bar{p}, \quad \sigma \nmid \bar{r}, \quad \text{and} \quad p, r \in \mathcal{C}^u.$$

Therefore $\bar{r} = \bar{p}$ by Lemma 2.18. But then

$$\bar{p} = \bar{r} \in \bar{\tau}\psi\left(e_j A e_j\right),$$

contrary to (47).

(ii) Now suppose $A$ is non-cancellative. We claim that there exists a cycle $p$ as in Claim (i). Indeed, $R \neq S$ by Theorem 4.41. Assume to the contrary that for each cycle $p$ satisfying $\bar{p} \notin R$, we have $\sigma \mid \bar{p}$. Then by the contrapositive of this assumption, for each cycle $q$ satisfying $\sigma \nmid \bar{q}$, we have $\bar{q} \in R$. But $S$ is generated by $\sigma$ and a set of monomials in $B$ not divisible by $\sigma$, by Proposition 3.10. Therefore $S \subseteq R$ since $\sigma \in R$. Whence $S = R$, contrary to assumption. □

**Theorem 4.45.** *Let $A$ be a non-cancellative dimer algebra that admits a cyclic contraction. Then each algebra*

$$A, \; Z, \; \hat{Z}, \; R,$$

*is nonnoetherian.*

*Proof.* Let $\psi : A \to A'$ be a cyclic contraction.

(i) We first claim that $R$ is nonnoetherian. Indeed, since $A$ is non-cancellative, there is a cycle $p \in A$ such that for each $n \geq 1$,

$$\bar{p}^n \notin R$$

by Lemma 4.44. Whence there is some $N \geq 1$ such that for each $n \geq 1$,

$$\bar{p}^n \sigma^N \in \bar{\psi}(Z) \tag{49}$$

by Proposition 4.30.2. Therefore

$$\bar{p}^n \sigma^N \in R$$

by Theorem 4.27. Thus there is an ascending chain of ideals of $R$,

$$\left(\bar{p}\sigma^N\right) \subseteq \left(\bar{p}\sigma^N, \bar{p}^2\sigma^N\right) \subseteq \left(\bar{p}\sigma^N, \bar{p}^2\sigma^N, \bar{p}^3\sigma^N\right) \subseteq \cdots \tag{50}$$

Assume to the contrary that for some $\ell \geq 2$, there are elements $g_1, \ldots, g_{\ell-1} \in R$ such that

$$\bar{p}^\ell \sigma^N = \sum_{n=1}^{\ell-1} g_n \bar{p}^n \sigma^N.$$

Then since $R$ is a domain,

$$\bar{p}^\ell = \sum_{n=1}^{\ell-1} g_n \bar{p}^n.$$

Furthermore, since $R$ is generated by monomials in the polynomial ring $B$, there is some $1 \leq n \leq \ell - 1$ such that $g_n$ is a monomial and

$$\bar{p}^\ell = g_n \bar{p}^n.$$

Whence $g_n = \bar{p}^{\ell-n}$, again since $R$ is a domain. But then $\bar{p}^{\ell-n}$ is in $R$, a contradiction. Thus each inclusion in the chain (50) is proper. Therefore $R$ is nonnoetherian, proving our claim.

(ii) We now claim that $A$ and $Z$ are nonnoetherian. Again consider the elements (49) in $\tau(Z)$. Denote by $z_n \in Z$ the central element whose $\bar{\psi}$-image is $\bar{p}^n\sigma^N$. Consider the ascending chain of (two-sided) ideals of $A$,

$$\langle z_1 \rangle \subseteq \langle z_1, z_2 \rangle \subseteq \langle z_1, z_2, z_3 \rangle \subseteq \cdots . \tag{51}$$

Assume to the contrary that for some $\ell \geq 2$, there are elements $a_1, \ldots, a_{\ell-1} \in A$ such that

$$z_\ell = \sum_{n=1}^{\ell-1} a_n z_n. \tag{52}$$

Since $\bar{p} \notin R$, there is a vertex $i \in Q_0$ such that $\bar{p}$ is not in $\bar{\tau}\psi(e_iAe_i)$. From (52) we obtain

$$\bar{p}^\ell \sigma^N = \bar{\tau}\psi(z_\ell e_i) = \sum \bar{\tau}\psi(a_n e_i)\bar{p}^n\sigma^N.$$

Whence

$$\bar{p}^\ell = \sum \bar{\tau}\psi(a_n e_i)\bar{p}^n. \tag{53}$$

Furthermore, since each $z_n$ is central,

$$\sum (e_i a_n)(z_n e_i) = e_i z_\ell = z_\ell e_i = \sum (a_n e_i)(z_n e_i).$$

Thus each product $a_n e_i$ is in the corner ring $e_iAe_i$. Therefore each image $\bar{\tau}\psi(a_n e_i)$ is in $\bar{\tau}\psi(e_iAe_i)$. In particular, $\bar{\tau}\psi(a_n e_i)$ cannot equal $\bar{p}^{\ell-n}$ for any $\ell - n \geq 1$ by our choice of vertex $i$. But $R$ is generated by monomials in the polynomial ring $B$. It follows that (53) cannot hold. Thus each inclusion in the chain (51) is proper. Therefore $A$ is nonnoetherian.

Since the elements $z_1, z_2, z_3, \ldots$ are in $Z$, we may consider the ascending chain of ideals of $Z$,

$$(z_1) \subseteq (z_1, z_2) \subseteq (z_1, z_2, z_3) \subseteq \cdots .$$

A similar argument then shows that $Z$ is also nonnoetherian.

(iii) Finally, we claim that $\hat{Z}$ is nonnoetherian. For each $i \in Q_0$, there are no cycles in $(\operatorname{nil} Z)e_i$ by Theorem 4.24. Thus the cycle $z_n e_i$ is not in $(\operatorname{nil} Z)e_i$. Whence $z_n$ is not in $\operatorname{nil} Z$. The claim then follows similar to Claim (ii). □

Although $\hat{Z}$ and $R$ are nonnoetherian, we will show that they each have Krull dimension 3 and are generically noetherian (Theorem 4.66). Furthermore, we will show that the homotopy algebra $\tilde{A}$ of $A$ is also nonnoetherian (Theorem 4.65).

**Theorem 4.46.** *Let $A$ be a non-cancellative dimer algebra that admits a cyclic contraction. Then $A$ is an infinitely generated $Z$-module.*

*Proof.* Let $\psi : A \to A'$ be a cyclic contraction. Assume to the contrary that there are elements $a_1, \ldots, a_N \in A$ such that

$$A = \sum_{n=1}^{N} Z a_n.$$

It suffices to suppose that each $a_n$ is in some corner ring $e_j A e_i$, for otherwise we could instead consider the finite set

$$\{e_j a_n e_i \mid i, j \in Q_0,\ 1 \le n \le N\}$$

as a generating set for $A$ as a $Z$-module. For each cycle $p \in \mathcal{C}$, there is thus a subset $J_p \subseteq \{1, \ldots, N\}$ and nonzero central elements $z_n \in Z$ such that

$$p = \sum_{n \in J_p} z_n a_n. \tag{54}$$

Therefore by Theorem 4.27,

$$\bar{p} = \sum_{n \in J_p} \bar{\psi}(z_n) \bar{a}_n \in \sum_{n \in J_p} R \bar{a}_n. \tag{55}$$

Set

$$J := \bigcup_{p \in \mathcal{C}} J_p \subseteq \{1, \ldots, N\}.$$

Then (55) implies that

$$S \subseteq \sum_{n \in J} R \bar{a}_n. \tag{56}$$

Again consider (54), and suppose $n \in J_p$. Then $a_n$ is in $e_iAe_i$ since $p$ is in $e_iAe_i$ and $z_n$ is central. Whence $\bar{a}_n$ is in $S$. Thus

$$S \supseteq \sum_{n \in J} R\bar{a}_n.$$

Therefore, together with (56), we obtain

$$S = \sum_{n \in J} R\bar{a}_n. \tag{57}$$

In particular, $S$ is a finitely generated $R$-module.

But $R$ is an infinitely generated $k$-algebra by Theorem 4.45. Furthermore, $S$ is a finitely generated $k$-algebra by Theorems 3.3 and 3.5. Therefore $S$ is an infinitely generated $R$-module by the Artin-Tate lemma, in contradiction to (57). $\square$

We will show that the homotopy algebra $\tilde{A}$ is also an infinitely generated module over its center $R$ (Theorem 4.65).

Following Theorem 4.46, an immediate question is whether non-cancellative dimer algebras satisfy a polynomial identity.[8]

**Proposition 4.47.** *Suppose $\psi : A \to A'$ is a cyclic contraction. If the head (or tail) of each arrow in $Q_1^*$ has indegree 1, then $A$ contains a free subalgebra. In particular, $A$ is not PI.*

*Proof.* Let $a \in Q_1^*$; then the indegree of $\mathrm{h}(a)$ is 1 by assumption. Consider the paths $p, q$ and the path $b$ of maximal length such that $bap$ and $baq$ are unit cycles. Let $b' \in Q_1'$ be a leftmost arrow subpath of $b$. Since $b$ has maximal length, the indegree of $\mathrm{h}(b') = \mathrm{h}(b)$ is at least 2. In particular, $b'$ is not contracted. Furthermore, no vertex in $Q'$ has indegree 1 since $A'$ is cancellative. Whence

$$\psi(bap) = b'p \quad \text{and} \quad \psi(baq) = b'q$$

are unit cycles in $Q'$. Therefore $p, q$ is a non-cancellative pair and $\bar{p} = \bar{q}$.

Since $A'$ is cancellative, there is a simple matching $D \in \mathcal{S}'$ which contains $b'$, by Theorem 4.41. Furthermore, since $D$ is a simple matching, there is a path $s$ in $Q'$ from $\mathrm{h}(p)$ to $\mathrm{t}(p)$ which is supported on $Q' \setminus D$. In particular,

$$x_D \nmid \bar{\tau}(sp) = \bar{\tau}(sq).$$

Since the indegree of the head of each contracted arrow is 1, $\psi$ is surjective. Thus there is path $r$ in $Q$ from $\mathrm{h}(p)$ to $\mathrm{t}(p)$ satisfying $\psi(r) = s$. Whence

$$x_D \nmid \bar{\tau}(sp) = \overline{rp} = \overline{rq}.$$

[8]Thanks to Paul Smith and Toby Stafford for inquiring whether non-cancellative dimer algebras are PI.

But then $b$ is not a subpath of $rp$ or $rq$ (modulo $I$) since $x_D \mid \bar{b}'$. Consequently, there are no relations between the cycles $rp$ and $rq$. Therefore

$$k\langle rp, rq\rangle$$

is a free subalgebra of $A$. $\square$

**Example 4.48.** Consider the dimer algebras $A_1$ and $A_2$ with quivers $Q^1$ and $Q^2$ given in Figures 15.i.a and 15.ii.a respectively. Both $A_1$ and $A_2$ have free subalgebras, although only $A_1$ satisfies the hypotheses of Proposition 4.47. Indeed, let $a$ and $b$ be the red and blue arrows in $Q^2$. Then $k\langle a, b\rangle$ is a free subalgebra of $A_2$.

**Question 4.49.** Is there a non-cancellative dimer algebra that satisfies a polynomial identity?

**Theorem 4.50.** *Let $A$ be a dimer algebra which admits a cyclic contraction. Then the following are equivalent:*

*(1) $A$ is cancellative.*
*(2) $A$ is noetherian.*
*(3) $Z$ is noetherian.*
*(4) $A$ is a finitely generated $Z$-module.*
*(5) The vertex corner rings $e_iAe_i$ are pairwise isomorphic algebras.*
*(6) Each vertex corner ring $e_iAe_i$ is isomorphic to $Z$.*

*Proof.* First suppose $A$ is cancellative. Then $Z$ is noetherian and $A$ is a finitely generated $Z$-module by Theorem 3.3.3. Therefore $A$ is noetherian. Furthermore, the vertex corner rings are pairwise isomorphic and isomorphic to $Z$ by Theorem 3.3.1.

Conversely, suppose $A$ is non-cancellative. Then $A$ and $Z$ are nonnoetherian by Theorem 4.45; $A$ is an infinitely generated $Z$-module by Theorem 4.46; and the vertex corner rings are not all isomorphic by Theorem 4.41. $\square$

**Notation 4.51.** In the remainder of this section, we will identify $\hat{Z}$ with its isomorphic $\bar{\psi}$-image in $R$ (Theorem 4.27), and thus write $\hat{Z} \subseteq R$.

**Lemma 4.52.** *The cycle algebra $S$ is a finitely generated $k$-algebra and a normal domain of Krull dimension 3. Furthermore,* $\operatorname{Max} S$ *is a toric Gorenstein singularity.*

*Proof.* Since $A'$ is a cancellative dimer algebra, it is well known that its center $Z'$ is a finitely generated $k$-algebra (Theorem 3.3.3), and a normal toric Gorenstein domain (Corollary 4.28) of Krull dimension 3. Furthermore,

$$Z' \overset{\text{(I)}}{\cong} S' \overset{\text{(II)}}{=} S,$$

where (I) holds by Theorem 3.5, and (II) holds by our assumption that $\psi$ is cyclic. $\square$

**Lemma 4.53.** *The morphisms*

$$\begin{aligned} \kappa_Z &: \operatorname{Max} S \to \operatorname{Max} \hat{Z}, \qquad \mathfrak{n} \to \mathfrak{n} \cap \hat{Z}, \\ \kappa_R &: \operatorname{Max} S \to \operatorname{Max} R, \qquad \mathfrak{n} \to \mathfrak{n} \cap R, \end{aligned} \tag{58}$$

*and*

$$\begin{aligned} \iota_Z &: \operatorname{Spec} S \to \operatorname{Spec} \hat{Z}, \qquad \mathfrak{q} \to \mathfrak{q} \cap \hat{Z}, \\ \iota_R &: \operatorname{Spec} S \to \operatorname{Spec} R, \qquad \mathfrak{q} \to \mathfrak{q} \cap R, \end{aligned}$$

*are well-defined and surjective.*

*Proof.* Let $\mathfrak{n} \in \operatorname{Max} S$. By Lemma 4.52, $S$ is a finitely generated $k$-algebra, and by assumption $k$ is an algebraically closed field. Therefore the intersections $\mathfrak{n} \cap \hat{Z}$ and $\mathfrak{n} \cap R$ are maximal ideals of $\hat{Z}$ and $R$ respectively.

Surjectivity of $\kappa_Z$ (resp. $\kappa_R$) follows from Claim (iii) in the proof of Theorem 3.5, with $S$ in place of $B$, and $\hat{Z}$ (resp. $R$) in place of $\tau(Z)$.

By assumption $k$ is also uncountable. Surjectivity of $\iota_Z$ (resp. $\iota_R$) then follows from the surjectivity of $\kappa_Z$ (resp. $\kappa_R$), by [B2, Lemma 2.15]. □

**Lemma 4.54.** *If $\mathfrak{p} \in \operatorname{Spec} \hat{Z}$ contains a monomial, then $\mathfrak{p}$ contains $\sigma$.*

*Proof.* Suppose $\mathfrak{p}$ contains a monomial $g$. Since $\mathfrak{p}$ is a proper ideal, $g$ is not in $k$. Thus there is a non-vertex cycle $p$ such that $\bar{p} = g$.

Let $q^+$ be a path from $\mathrm{h}(p^+)$ to $\mathrm{t}(p^+)$. Then $(pq)^+$ is a cycle in $Q^+$. Thus there is some $n \geq 1$ such that $\bar{p}\bar{q} = \sigma^n$ by Lemma 4.11.1.

By Lemma 4.53, there is a prime ideal $\mathfrak{q} \in \operatorname{Max} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then $\bar{p}\bar{q} = \sigma^n$ is in $\mathfrak{n}$ since $\bar{q} \in S$ and $\bar{p} = g \in \mathfrak{p}$. Thus $\sigma$ is also in $\mathfrak{q}$ since $\mathfrak{q}$ is a prime ideal. Therefore by Lemma 1.5,

$$\sigma \in \mathfrak{q} \cap \hat{Z} = \mathfrak{p}.$$

□

Denote the origin of $\operatorname{Max} S$ by

$$\mathfrak{n}_0 := (\bar{s} \in S \mid s \text{ a non-vertex cycle})\, S \in \operatorname{Max} S.$$

Consider the maximal ideals of $\hat{Z}$ and $R$ respectively,

$$\mathfrak{z}_0 := \mathfrak{n}_0 \cap \hat{Z} \quad \text{and} \quad \mathfrak{m}_0 := \mathfrak{n}_0 \cap R.$$

**Lemma 4.55.** *The localizations $\hat{Z}_{\mathfrak{z}_0}$ and $R_{\mathfrak{m}_0}$ are nonnoetherian.*

*Proof.* Let $\bar{p} \in S \setminus \hat{Z}$ be as in Claim (i) in the proof of Theorem 4.45. Since $\hat{Z}$ is generated by monomials in the polynomial ring $B$, the monomial $\bar{p}^n$ is not in the localization $\hat{Z}_{\mathfrak{z}_0}$ for any $n \geq 1$. Whence the chain (50) does not terminate in $\hat{Z}_{\mathfrak{z}_0}$. Therefore $\hat{Z}_{\mathfrak{z}_0}$ is nonnoetherian. Similarly $R_{\mathfrak{m}_0}$ is nonnoetherian. □

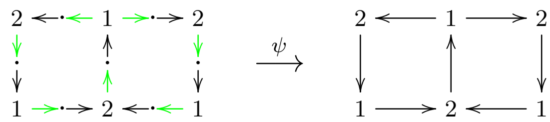

FIGURE 27. An example where $\sigma$ divides all non-constant monomials in $R$. The quivers are drawn on a torus, and the contracted arrows are drawn in green.

**Lemma 4.56.** *Suppose that each non-constant monomial in $\hat{Z}$ is divisible (in $B$) by $\sigma$. If $\mathfrak{p} \in \operatorname{Spec} \hat{Z}$ contains a monomial, then $\mathfrak{p} = \mathfrak{z}_0$.*

*Proof.* Suppose $\mathfrak{p} \in \operatorname{Spec} \hat{Z}$ contains a monomial. Then $\sigma$ is in $\mathfrak{p}$ by Lemma 4.54. Furthermore, there is some $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap \hat{Z} = \mathfrak{p}$ by Lemma 4.53.

Suppose $f$ is a non-constant monomial in $\hat{Z}$. Then by assumption, there is a monomial $h$ in $B$ such that $f = \sigma h$. By Lemma 4.29, $h$ is also in $S$. Therefore $f = \sigma h \in \mathfrak{q}$ since $\sigma \in \mathfrak{p} \subseteq \mathfrak{q}$. But $f \in \hat{Z}$. Whence

$$f \in \mathfrak{q} \cap \hat{Z} = \mathfrak{p}.$$

Since $f$ was arbitrary, $\mathfrak{p}$ contains all non-constant monomials in $\hat{Z}$. $\square$

**Remark 4.57.** In Lemma 4.56, we assumed $\sigma$ divides all non-constant monomials in $\hat{Z}$. An example of a dimer algebra with this property is the dimer algebra with quiver given in Figure 27. In this example, $R = k + \sigma S$ and $S = k[xz, xw, yz, yw]$ (with impression given by the arrow orientations in Figure 14).

**Lemma 4.58.** *Suppose that there is a non-constant monomial in $\hat{Z}$ which is not divisible (in $B$) by $\sigma$. Let $\mathfrak{m} \in \operatorname{Max} \hat{Z} \setminus \{\mathfrak{z}_0\}$. Then there is a non-constant monomial $g \in \hat{Z}$ such that*

$$g \notin \mathfrak{m} \quad \textit{and} \quad \sigma \nmid g.$$

*Proof.* Let $\mathfrak{m} \in \operatorname{Max} \hat{Z} \setminus \{\mathfrak{z}_0\}$.

(i) We first claim that there is a non-constant monomial in $\hat{Z}$ which is not in $\mathfrak{m}$. Assume otherwise. Then

$$\mathfrak{n}_0 \cap \hat{Z} \subseteq \mathfrak{m}.$$

But $\mathfrak{n}_0 \cap \hat{Z}$ is a maximal ideal by Lemma 4.53. Thus $\mathfrak{z}_0 = \mathfrak{n}_0 \cap \hat{Z} = \mathfrak{m}$, a contradiction.

(ii) We now claim that there is a non-constant monomial in $\hat{Z} \setminus \mathfrak{m}$ which is not divisible by $\sigma$. Assume otherwise; that is, assume that every non-constant monomial in $\hat{Z}$, which is not divisible by $\sigma$, is in $\mathfrak{m}$. By assumption, $\sigma$ does not divide all non-constant monomials in $\hat{Z}$. Thus there is at least one monomial in $\mathfrak{m}$. Therefore $\sigma$ is in $\mathfrak{m}$ by Lemma 4.54.

By Lemma 4.53, there is an $\mathfrak{n} \in \operatorname{Max} S$ such that $\mathfrak{n} \cap \hat{Z} = \mathfrak{m}$. Then $\sigma \in \mathfrak{n}$ since $\sigma \in \mathfrak{m}$. Suppose $\sigma$ divides the monomial $g \in \hat{Z}$; say $g = \sigma h$ for some monomial $h \in B$. Then $h \in S$ by Lemma 4.29. Thus $g = \sigma h \in \mathfrak{n}$. Whence

$$g \in \mathfrak{n} \cap \hat{Z} = \mathfrak{m}.$$

Thus every non-constant monomial in $\hat{Z}$, which *is* divisible by $\sigma$, is also in $\mathfrak{m}$. Therefore every non-constant monomial in $\hat{Z}$ is in $\mathfrak{m}$. But this contradicts our choice of $\mathfrak{m}$ by Claim (i). $\square$

Recall the subsets (9) of $\operatorname{Max} S$ and the morphisms (58). For brevity, we will write $U_Z$, $U_Z^*$, $U_R$, $U_R^*$ for the respective subsets $U_{\hat{Z},S}$, $U^*_{\hat{Z},S}$, $U_{R,S}$, $U^*_{R,S}$. Furthermore, we will denote their respective complements with a superscript $c$.

**Proposition 4.59.** *Let* $\mathfrak{n} \in \operatorname{Max} S$. *Then*

$$\mathfrak{n} \cap \hat{Z} \neq \mathfrak{z}_0 \quad \text{if and only if} \quad \hat{Z}_{\mathfrak{n} \cap \hat{Z}} = S_{\mathfrak{n}}. \tag{59}$$

*Consequently,*

$$\kappa_Z(U_Z) = \operatorname{Max} \hat{Z} \setminus \{\mathfrak{z}_0\}.$$

*Furthermore,*

$$\kappa_R(U_R) = \operatorname{Max} R \setminus \{\mathfrak{m}_0\}.$$

*Proof.* (i) Let $\mathfrak{n} \in \operatorname{Max} S$ and set $\mathfrak{m} := \mathfrak{n} \cap \hat{Z}$. We first want to show that if $\mathfrak{m} \neq \mathfrak{z}_0$, then $\hat{Z}_{\mathfrak{m}} = S_{\mathfrak{n}}$.

Consider $g \in S \setminus \hat{Z}$. It suffices to show that $g$ is in the localization $\hat{Z}_{\mathfrak{m}}$. But $S$ is generated by $\sigma$ and a set of monomials in $B$ not divisible by $\sigma$, by Proposition 3.10. Furthermore, $\sigma$ is in $\hat{Z}$ by Lemma 1.5. Thus it suffices to suppose $g$ is a non-constant monomial which is not divisible by $\sigma$. Let $u \in \mathbb{Z}^2$ and $p \in \mathcal{C}^u$ be such that $\bar{p} = g$.

We claim that $u \neq 0$. Indeed, suppose otherwise. Then $p^+$ is a cycle in $Q^+$. Thus $\bar{p} = \sigma^n$ for some $n \geq 1$ by Lemma 4.11.1. But $\sigma^n$ is in $\hat{Z}$. Consequently, $\bar{p} = g$ is in $\hat{Z}$, contrary to our choice of $g$. Therefore $u \neq 0$.

(i.a) First suppose $\sigma$ does not divide all non-constant monomials in $\hat{Z}$. Fix $i \in Q_0$. By Lemma 4.58, there is a non-vertex cycle $q \in e_i A e_i$ such that

$$\bar{q} \in \hat{Z} \setminus \mathfrak{m} \quad \text{and} \quad \sigma \nmid \bar{q}.$$

Let $v \in \mathbb{Z}^2$ be such that $q \in \mathcal{C}^v$. Then $v \neq 0$ since $\sigma \nmid \bar{q}$.

We claim that $u \neq v$. Assume to the contrary that $u = v$. Then by Lemma 2.3.1, $\bar{p} = \bar{q}$ since $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$. But $\bar{q}$ is in $\hat{Z}$, whereas $\bar{p}$ is not, a contradiction. Therefore $u \neq v$.

Since $u \neq v$ are nonzero, the paths $p^+$ and $q^+$ are transverse in $Q^+$. Thus $p^+$ and $q^+$ intersect at a vertex $j^+$. Therefore $p$ and $q$ factor into the product of paths

$$p = p_2 e_j p_1 \quad \text{and} \quad q = q_2 e_j q_1.$$

Consider the cycle

$$r := q_2 p_1 p_2 q_1 \in e_i A e_i.$$

Let $\ell \geq 0$ be such that $\sigma^\ell \mid \bar{r}$ and $\sigma^{\ell+1} \nmid \bar{r}$. Since $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$, there is a cyclic subpath $r_1^+$ of $(p_1p_2q_1)^+$ that is not a subpath of $(p_1p_2)^+$ or $q_1^+$, or a cyclic subpath $r_2^+$ of $(q_2p_1p_2)^+$ that is not a subpath of $q_2^+$ or $(p_1p_2)^+$, such that

$$\bar{r}_1 = \sigma^{m_1} \quad \text{and} \quad \bar{r}_2 = \sigma^{m_2},$$

where $m_1 + m_2 = \ell$. Consider the cycle $t$ obtained from $r$ by removing the cyclic subpaths $r_1$ and $r_2$, modulo $I$. Then $\sigma \nmid \bar{t}$.

Since $i \in Q_0$ was arbitrary, we have $\bar{t} \in R$. But $\sigma \nmid \bar{t}$. Thus by Proposition 4.30.1.,

$$\bar{t} \in \hat{Z}.$$

Therefore, since $\bar{q} \in \hat{Z} \setminus \mathfrak{m}$,

$$g = \bar{p} = \bar{r}\bar{q}^{-1} = \sigma^\ell \bar{t}\bar{q}^{-1} \in \hat{Z}_{\mathfrak{m}}.$$

But $g$ was an arbitrary non-constant monomial. Thus, since $S$ is generated by monomials,

$$S \subseteq \hat{Z}_{\mathfrak{m}}.$$

Therefore[9]

$$S_{\mathfrak{n}} = \hat{Z}_{\mathfrak{m}}.$$

(i.b) Now suppose $\sigma$ divides all non-constant monomials in $\hat{Z}$. Further suppose $\mathfrak{m} \neq \mathfrak{z}_0$. Then by Lemma 4.56, $\mathfrak{m}$ does not contain any monomials. In particular, $\sigma \notin \mathfrak{m}$. By Lemma 4.30.2, there is an $N \geq 0$ such that $g\sigma^N \in \hat{Z}$. Thus

$$g = (g\sigma^N)\sigma^{-N} \in \hat{Z}_{\mathfrak{m}}.$$

But $g$ was an arbitrary non-constant monomial. Therefore

$$S \subseteq \hat{Z}_{\mathfrak{m}}.$$

It follows that

$$S_{\mathfrak{n}} = \hat{Z}_{\mathfrak{m}}.$$

Therefore, in either case (a) or (b), Claim (i) holds.

(ii) Now suppose $\mathfrak{n} \in \operatorname{Max} S$ satisfies $\mathfrak{n} \cap R \neq \mathfrak{m}_0$. We claim that $R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}$.

Since $\mathfrak{n} \cap R \neq \mathfrak{m}_0$, there is a monomial $g \in S \setminus \mathfrak{n}$. By Proposition 4.30.3, there is some $N \geq 1$ such that $g^N \in \hat{Z}$. But $g^N \notin \mathfrak{n}$ since $\mathfrak{n}$ is a prime ideal. Thus

$$g^N \in \hat{Z} \setminus (\mathfrak{n} \cap \hat{Z}).$$

Whence

$$\mathfrak{n} \cap \hat{Z} \neq \mathfrak{z}_0.$$

[9] Denote by $\tilde{\mathfrak{m}} := \mathfrak{m}\hat{Z}_{\mathfrak{m}}$ the maximal ideal of $\hat{Z}_{\mathfrak{m}}$. Then, since $\hat{Z} \subset S$,

$$\hat{Z}_{\mathfrak{m}} = \hat{Z}_{\tilde{\mathfrak{m}}\cap\hat{Z}} \subseteq S_{\tilde{\mathfrak{m}}\cap S} \subseteq (\hat{Z}_{\mathfrak{m}})_{\tilde{\mathfrak{m}}\cap\hat{Z}_{\mathfrak{m}}} = \hat{Z}_{\mathfrak{m}}.$$

Therefore

$$S_{\mathfrak{n}} \overset{(\text{I})}{=} \hat{Z}_{\mathfrak{n}\cap\hat{Z}} \overset{(\text{II})}{\subseteq} R_{\mathfrak{n}\cap R} \subseteq S_{\mathfrak{n}},$$

where (I) holds by Claim (i), and (II) follows from Theorem 4.27. Consequently $R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}$, proving our claim.

(iii) Finally, we claim that

$$\hat{Z}_{\mathfrak{z}_0} \neq S_{\mathfrak{n}_0} \quad \text{and} \quad R_{\mathfrak{m}_0} \neq S_{\mathfrak{n}_0}.$$

These inequalities hold since the local algebras $\hat{Z}_{\mathfrak{z}_0}$ and $R_{\mathfrak{m}_0}$ are nonnoetherian by Lemma 4.55, whereas $S_{\mathfrak{n}}$ is noetherian by Lemma 4.52. □

**Lemma 4.60.** *Let $\mathfrak{q}$ and $\mathfrak{q}'$ be prime ideals of $S$. Then*

$$\mathfrak{q} \cap \hat{Z} = \mathfrak{q}' \cap \hat{Z} \quad \textit{if and only if} \quad \mathfrak{q} \cap R = \mathfrak{q}' \cap R.$$

*Proof.* (i) Suppose $\mathfrak{q} \cap \hat{Z} = \mathfrak{q}' \cap \hat{Z}$, and let $s \in \mathfrak{n} \cap R$. Then $s \in R$. Whence there is some $n \geq 1$ such that $s^n \in \hat{Z}$ by Lemma 4.30.3. Thus

$$s^n \in \mathfrak{q} \cap \hat{Z} = \mathfrak{q}' \cap \hat{Z}.$$

Therefore $s^n \in \mathfrak{n}'$. Thus $s \in \mathfrak{q}'$ since $\mathfrak{q}'$ is prime. Consequently $s \in \mathfrak{q}' \cap R$. Therefore $\mathfrak{q} \cap R \subseteq \mathfrak{q}' \cap R$. Similarly $\mathfrak{q} \cap R \supseteq \mathfrak{q}' \cap R$.

(ii) Now suppose $\mathfrak{q} \cap R = \mathfrak{q}' \cap R$, and let $s \in \mathfrak{q} \cap \hat{Z}$. Then $s \in \hat{Z} \subseteq R$. Thus

$$s \in \mathfrak{q} \cap R = \mathfrak{q}' \cap R.$$

Whence $s \in \mathfrak{q}' \cap \hat{Z}$. Therefore $\mathfrak{q} \cap \hat{Z} \subseteq \mathfrak{q}' \cap \hat{Z}$. Similarly $\mathfrak{q} \cap \hat{Z} \supseteq \mathfrak{q}' \cap \hat{Z}$. □

**Proposition 4.61.** *The subsets $U_Z$ and $U_R$ of $\operatorname{Max} S$ coincide,*

$$U_Z = U_R.$$

*Proof.* (i) We first claim that

$$U_Z \subseteq U_R.$$

Indeed, suppose $\mathfrak{n} \in U_Z$. Then since $\hat{Z} \subseteq R \subset S$,

$$S_{\mathfrak{n}} = \hat{Z}_{\mathfrak{n}\cap\hat{Z}} \subseteq R_{\mathfrak{n}\cap R} \subseteq S_{\mathfrak{n}}.$$

Thus

$$R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}.$$

Therefore $\mathfrak{n} \in U_R$, proving our claim.

(ii) We now claim that

$$U_R \subseteq U_Z.$$

Let $\mathfrak{n} \in U_R$. Then $R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}$. Thus by Proposition 4.59,

$$\mathfrak{n} \cap R \neq \mathfrak{n}_0 \cap R.$$

Therefore by Lemma 4.60,

$$\mathfrak{n} \cap \hat{Z} \neq \mathfrak{n}_0 \cap \hat{Z}.$$

But then again by Proposition 4.59,

$$\hat{Z}_{\mathfrak{n}\cap\hat{Z}} = S_{\mathfrak{n}}.$$

Whence $\mathfrak{n} \in U_Z$, proving our claim. $\square$

**Definition 4.62.** We say an integral domain $R$ is *generically noetherian* if there is an open dense set $W \subset \operatorname{Max} R$ such that for each $\mathfrak{m} \in W$, the localization $R_{\mathfrak{m}}$ is noetherian.

**Theorem 4.63.** *The following subsets of* $\operatorname{Max} S$ *are nonempty and coincide:*

$$\begin{gathered} U_Z^* = U_Z = U_R^* = U_R \\ = \kappa_Z^{-1}(\operatorname{Max}\hat{Z} \setminus \{\mathfrak{z}_0\}) = \kappa_R^{-1}\left(\operatorname{Max} R \setminus \{\mathfrak{m}_0\}\right). \end{gathered} \tag{60}$$

*In particular, $\hat{Z}$ and $R$ are isolated nonnoetherian singularities and generically noetherian.*

*Proof.* The equalities (60) hold since:

- By Proposition 4.61, $U_R = U_Z$.
- By (59) in Proposition 4.59,
$$U_Z = \kappa_Z^{-1}\left(\operatorname{Max}\hat{Z} \setminus \{\mathfrak{z}_0\}\right) \quad \text{and} \quad U_R = \kappa_R^{-1}\left(\operatorname{Max} R \setminus \{\mathfrak{m}_0\}\right).$$
- By Lemma 4.52, the localization $S_{\mathfrak{n}}$ is noetherian for each $\mathfrak{n} \in \operatorname{Max} S$. Therefore again by (59),
$$U_Z^* = U_Z \quad \text{and} \quad U_R^* = U_R.$$

Furthermore, $\kappa_R^{-1}\left(\operatorname{Max} R \setminus \{\mathfrak{m}_0\}\right)$ is nonempty since there is a maximal ideal of $R$ distinct from $\mathfrak{m}_0$, and $\kappa_R$ is surjective by Lemma 4.53. $\square$

**Proposition 4.64.** *The locus $U_R \subset \operatorname{Max} S$ is an open set.*

*Proof.* We claim that the complement of $U_R \subset \operatorname{Max} S$ is the closed subvariety

$$U_R^c = \{\mathfrak{n} \in \operatorname{Max} S \mid \mathfrak{n} \supseteq \mathfrak{m}_0 S\} =: \mathcal{Z}(\mathfrak{m}_0 S).$$

Indeed, let $\mathfrak{n} \in \operatorname{Max} S$. First suppose $\mathfrak{m}_0 S \subseteq \mathfrak{n}$. Then

$$\mathfrak{m}_0 \subseteq \mathfrak{m}_0 S \cap R \subseteq \mathfrak{n} \cap R. \tag{61}$$

Whence $\mathfrak{n} \cap R = \mathfrak{m}_0$ since $\mathfrak{m}_0$ is a maximal ideal of $R$. Thus $\mathfrak{n} \notin U_R$ by Theorem 4.63. Therefore $U_R^c \supseteq \mathcal{Z}(\mathfrak{m}_0 S)$.

Conversely, suppose $\mathfrak{n} \notin U_R$. Assume to the contrary that $\mathfrak{m}_0 S \subseteq \mathfrak{n}$. Then (61) implies that $\mathfrak{n} \cap R = \mathfrak{m}_0$, contrary to Theorem 4.63. Therefore $U_R^c \subseteq \mathcal{Z}(\mathfrak{m}_0 S)$. $\square$

**Theorem 4.65.** *The homotopy algebra $\tilde{A}$ of $A$ is nonnoetherian and an infinitely generated module over its center $R$.*

*Proof.* The following hold:

- $(\tilde{\tau}, B)$ is an impression of $\tilde{A}$ by Theorem 4.35.
- $U_R^* = U_R$ by Theorem 4.63.
- $R \neq S$ by Theorem 4.41.

Therefore $\tilde{A}$ is nonnoetherian and an infinitely generated $R$-module by [B2, Theorem 3.2.2]. $\square$

**Theorem 4.66.** *Let $A$ be a non-cancellative dimer algebra and $\psi : A \to A'$ a cyclic contraction. Then the center $Z$, reduced center $\hat{Z}$, and homotopy center $R$ each have Krull dimension 3,*

$$\dim Z = \dim \hat{Z} = \dim R = \dim S = 3.$$

*Furthermore, the fraction fields of $\hat{Z}$, $R$, and $S$ coincide,*

$$\operatorname{Frac} \hat{Z} = \operatorname{Frac} R = \operatorname{Frac} S. \tag{62}$$

*Proof.* Recall that $\hat{Z}$, $R$, and $S$ are domains by Corollary 4.28. Furthermore, the subsets $U_Z$ and $U_R$ of $\operatorname{Max} S$ are nonempty by Theorem 4.63.

Since $U_Z$ and $U_R$ are nonempty, the equalities (62) hold by [B2, Lemma 2.4]. Furthermore,

$$\dim \hat{Z} = \dim S = \dim R$$

by [B2, Theorem 2.5.4]. Therefore $\hat{Z}$ and $R$ each have Krull dimension 3 by Lemma 4.52. Finally, let $\mathfrak{p} \in \operatorname{Spec} Z$. Then $\operatorname{nil} Z \subseteq \mathfrak{p}$. Therefore $\dim Z = \dim \hat{Z}$. $\square$

**Lemma 4.67.** *Let $g, h \in S$ be non-constant monomials such that $\sigma \nmid gh$. If $gh \notin R$, then $g \notin R$ and $h \notin R$.*

*Proof.* Suppose the hypotheses hold, and assume to the contrary that $g \in R$. Fix $i \in Q_0$. Since $g$ is a non-constant monomial in $R$ and $h$ is a non-constant monomial in $S$, there is some $u, v \in \mathbb{Z}^2$ and cycles

$$p \in \mathcal{C}_i^u \quad \text{and} \quad q \in \mathcal{C}^v$$

such that

$$\bar{p} = g \quad \text{and} \quad \bar{q} = h.$$

By assumption $\sigma \nmid gh = \bar{p}\bar{q}$. Thus $\sigma \nmid \bar{p}$ and $\sigma \nmid \bar{q}$. Therefore $u \neq 0$ and $v \neq 0$ by Lemma 4.11.1.

If $u = v$, then $\bar{p} = \bar{q}$ by Lemma 2.18. Whence $\bar{q} \in R$ since $\bar{p} \in R$. But then $\bar{p}\bar{q} \in R$, contrary to assumption. Therefore $u \neq v$. It follows that the lifts $p^+$ and $q^+$ are transverse cycles in $Q^+$. Thus there is a vertex $j^+ \in Q_0^+$ where $p^+$ and $q^+$ intersect. We may therefore write $p$ and $q$ as products of cycles

$$p = p_2 e_j p_1 \quad \text{and} \quad q = q_2 e_j q_1.$$

Consider the cycle

$$r = p_2 q_1 q_2 p_1 \in e_i A e_i.$$

Then

$$\bar{r} = \bar{p}\bar{q} = gh.$$

Therefore, since $i \in Q_0$ was arbitrary, $\bar{r} = gh$ is in $R$, a contradiction. $\square$

Recall that the reduction $X_{\text{red}}$ of a scheme $X$, that is, its reduced induced scheme structure, is the closed subspace of $X$ associated to the sheaf of ideals $\mathcal{I}$, where for each open set $U \subset X$,

$$\mathcal{I}(U) := \{ f \in \mathcal{O}_X(U) \mid f(\mathfrak{p}) = 0 \ \text{ for all } \ \mathfrak{p} \in U \}.$$

$X_{\text{red}}$ is the unique reduced scheme whose underlying topological space equals that of $X$. If $R := \mathcal{O}_X(X)$, then $\mathcal{O}_{X_{\text{red}}}(X_{\text{red}}) = R/\operatorname{nil} R$, where $\operatorname{nil} R$ is the nilradical of $R$ (that is, the radical of the zero ideal of $R$).

**Theorem 4.68.** *Let $A$ be a non-cancellative dimer algebra, $\psi : A \to A'$ a cyclic contraction, and $\tilde{A}$ the homotopy algebra of $A$.*

1. *The reduced center $\hat{Z}$ and homotopy center $R$ of $A$ are both depicted by the center $Z' \cong S$ of $A'$.*
2. *The reduced induced scheme structure of $\operatorname{Spec} Z$ and the scheme $\operatorname{Spec} R$ are birational to the noetherian scheme $\operatorname{Spec} S$, and each contain precisely one closed point of positive geometric dimension.*
3. *The spectra $\operatorname{Max} \hat{Z}$ and $\operatorname{Max} R$ may both be viewed as the Gorenstein algebraic variety $\operatorname{Max} S$ with the subvariety $U_R^c$ identified as a single 'smeared-out' point.*

*Proof.* (1) We first claim that $\hat{Z}$ and $R$ are depicted by $S$. By Theorem 4.63,

$$U_Z^* = U_Z \neq \emptyset \quad \text{and} \quad U_R^* = U_R \neq \emptyset.$$

Furthermore, by Lemma 4.53, the morphisms $\iota_Z$ and $\iota_R$ are surjective.[10]

(2.i) Claim (1) and [B2, Theorem 2.5.3] together imply that the schemes $\operatorname{Spec} \hat{Z}$ and $\operatorname{Spec} R$ are birational to $\operatorname{Spec} S$, and are isomorphic on $U_Z = U_R$. By Lemma 4.52, $S$ is a normal toric Gorenstein domain. By Theorem 4.63, $\operatorname{Max} \hat{Z}$ and $\operatorname{Max} R$ each contain precisely one point where the localizations of $\hat{Z}$ and $R$ are nonnoetherian, namely $\mathfrak{z}_0$ and $\mathfrak{m}_0$.

(2.ii) We claim that the closed points $\mathfrak{z}_0 \in \operatorname{Spec} \hat{Z}$ and $\mathfrak{m}_0 \in \operatorname{Spec} R$ have positive geometric dimension.

Indeed, since $A$ is non-cancellative, there is a cycle $p$ such that $\sigma \nmid \bar{p}$ and $\bar{p}^n \in S \setminus R$ for each $n \geq 1$, by Lemma 4.44. In particular, $\bar{p}$ is not a product $\bar{p} = gh$, where $g \in R$ or $h \in R$, by Lemma 4.67. Therefore

$$\bar{p} \notin \mathfrak{m}_0 S.$$

Thus for each $c \in k$, there is a maximal ideal $\mathfrak{n}_c \in \operatorname{Max} S$ such that

$$(\bar{p} - c, \mathfrak{m}_0)S \subseteq \mathfrak{n}_c.$$

[10]The fact that $S$ is a depiction of $R$ also follows from [B2, Theorem 3.2.1], since $(\tilde{\tau}, B)$ is an impression of $\tilde{A}$ by Theorem 4.35.

Consequently,

$$\mathfrak{m}_0 \subseteq (\bar{p} - c, \mathfrak{m}_0)S \cap R \subseteq \mathfrak{n}_c \cap R.$$

Whence $\mathfrak{n}_c \cap R = \mathfrak{m}_0$ since $\mathfrak{m}_0$ is maximal. Therefore by Theorem 4.63,

$$\mathfrak{n}_c \in U_R^c.$$

Set

$$\mathfrak{q} := \bigcap_{c \in k} \mathfrak{n}_c.$$

Then $\mathfrak{q}$ is a radical ideal since it is the intersection of radical ideals. Thus, since $S$ is noetherian, the Lasker-Noether theorem implies that there are minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell \in \operatorname{Spec} S$ over $\mathfrak{q}$ such that

$$\mathfrak{q} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_\ell.$$

Since $\ell < \infty$, at least one $\mathfrak{q}_i$ is a non-maximal prime, say $\mathfrak{q}_1$. Then

$$\mathfrak{m}_0 = \bigcap_{c \in k} (\mathfrak{n}_c \cap R) = \bigcap_{c \in k} \mathfrak{n}_c \cap R = \mathfrak{q} \cap R \subseteq \mathfrak{q}_1 \cap R.$$

Whence $\mathfrak{q}_1 \cap R = \mathfrak{m}_0$ since $\mathfrak{m}_0$ is maximal.

Since $\mathfrak{q}_1$ is a non-maximal prime ideal of $S$,

$$\operatorname{ht}(\mathfrak{q}_1) < \dim S.$$

Furthermore, $S$ is a depiction of $R$ by Claim (1). Thus

$$\operatorname{ght}(\mathfrak{m}_0) \leq \operatorname{ht}(\mathfrak{q}_1) < \dim S \stackrel{(\mathrm{I})}{=} \dim R,$$

where (I) holds by Theorem 4.66. Therefore

$$\operatorname{gdim} \mathfrak{m}_0 = \dim R - \operatorname{ght}(\mathfrak{m}_0) \geq 1,$$

proving our claim.

(3) Follows from Claims (1), (2), and Theorem 4.63. The locus $U_R^c$ is a (closed) subvariety by Proposition 4.64. □

**Remark 4.69.** Although $\hat{Z}$ and $R$ determine the same nonlocal variety using depictions, their associated affine schemes

$$\left(\operatorname{Spec} \hat{Z}, \mathcal{O}_{\hat{Z}}\right) \quad \text{and} \quad (\operatorname{Spec} R, \mathcal{O}_R)$$

will not be isomorphic if their rings of global sections, $\hat{Z}$ and $R$, are not isomorphic.

4.6. **Integral closure.** It is well known that the center of a cancellative dimer algebra is normal. In this section, we characterize the central normality of non-cancellative dimer algebras.

Throughout, $A$ is a non-cancellative dimer algebra and $\psi : A \to A'$ is a cyclic contraction. Let $\tilde{A}$ and $R = Z(\tilde{A})$ be the homotopy algebra and homotopy center of $A$, respectively. We denote by $\hat{Z}^c$ and $R^c$ the respective integral closures of $\hat{Z}$ and $R$.

**Proposition 4.70.** *The homotopy center $R$ is normal if and only if $\sigma S \subset R$.*

*Proof.* (1) First suppose $\sigma S \subset R$.

(1.i) By Lemma 4.52, $S$ is normal. Therefore, since $R$ is a subalgebra of $S$,

$$R^c \subseteq S. \tag{63}$$

(1.ii) Now let $s \in S \setminus R$. We claim that $s$ is not in $R^c$. Indeed, assume otherwise. Since $S$ is generated by monomials in the polynomial ring $B$, there are monomials $s_1, \ldots, s_\ell \in S$ such that

$$s = s_1 + \cdots + s_\ell.$$

Since $s \notin R$, there is some $1 \le k \le \ell$ such that $s_k \notin R$. Choose $s_k$ to have maximal degree among the subset of monomials in $\{s_1, \ldots, s_\ell\}$ which are not in $R$.

By assumption, $\sigma S \subset R$. Thus

$$\sigma \nmid s_k.$$

Since $s \in R^c$, there is some $n \geq 1$ and $r_0, \ldots, r_{n-1} \in R$ such that

$$s^n + r_{n-1}s^{n-1} + \cdots + r_1 s = -r_0 \in R. \tag{64}$$

By Lemma 4.44, the summand $s_k^n$ of $s^n$ is not in $R$ since $\sigma \nmid s_k$. Thus $-s_k^n$ is a summand of the left-hand side of (64). In particular, for some $1 \le m \le n$, there are monomial summands $r'$ of $r_m$ and $s' = s_{j_1} \cdots s_{j_m}$ of $s^m$, and a nonzero scalar $c \in k$, such that

$$r's' = cs_k^n.$$

By Lemma 4.67, $r'$ is a nonzero scalar since $r' \in R$, $s_k^n \notin R$, and $\sigma \nmid s_k^n$. Furthermore, $s'$ is a non-constant monomial since $r' \in R$ and $s_k^n \notin R$. Therefore

$$s_{j_1} \cdots s_{j_m} = s' = (c/r')s_k^n.$$

By Lemma 4.67, each monomial factor $s_{j_1}, \ldots, s_{j_m}$ is not in $R$. But the monomial $s_k$ was chosen to have maximal degree, a contradiction. Therefore

$$R^c \cap S = R. \tag{65}$$

It follows from (63) and (65) that

$$R^c = R^c \cap S = R.$$

(2) Now suppose $\sigma S \not\subset R$. Then there are monomials $s \in S \setminus R$ and $t \in S$ such that $s = t\sigma$. Let $n \geq 2$ be sufficiently large so that the product of $n$ unit cycles $\sigma_i^n \in A$ is equal (modulo $I$) to a cycle that contains each vertex in $Q$. Then

$$\sigma^n S \subset R.$$

In particular, the product $s^n = t^n \sigma^n$ is in $R$. But then $s \in \operatorname{Frac} S \setminus R$ is a root of the monic polynomial

$$x^n - s^n \in R[x].$$

Thus $s$ is in $R^c \setminus R$. Therefore $R$ is not normal. □

**Corollary 4.71.**

*(1) If the head or tail of each contracted arrow has indegree 1, then $R$ is normal.*
*(2) If $\psi$ contracts precisely one arrow, then $R$ is normal.*

*Proof.* In both cases (1) and (2), clearly $\sigma S \subset R$. □

**Proposition 4.72.** *For each $n \geq 1$, there are dimer algebras for which*

$$\sigma^n S \not\subset R \quad \textit{and} \quad \sigma^{n+1} S \subset R.$$

*Consequently, there are dimer algebras for which $R$ is not normal.*

*Proof.* Recall the conifold quiver $Q$ with one nested square given in Figure 21.i. Clearly $\sigma S \subset R$. More generally, the conifold quiver with $n \geq 1$ nested squares satisfies

$$\sigma^{n-1} S \not\subset R \quad \text{and} \quad \sigma^n S \subset R.$$

The corresponding homotopy center $R$ is therefore not normal for $n \geq 2$ by Proposition 4.70. □

Let $\tilde{\mathfrak{m}}_0 \subset \mathfrak{m}_0$ be the ideal of $R$ generated by all non-constant monomials in $R$ which are not powers of $\sigma$.

**Proposition 4.73.** *Let $n \geq 0$, and suppose $\sigma^n S \not\subset R$ and $\sigma^{n+1} S \subset R$. Then*

$$R = k[\sigma] + (\tilde{\mathfrak{m}}_0, \sigma^{n+1})S. \tag{66}$$

*Proof.* Suppose $p \in \mathcal{C}^u$ is a non-vertex cycle such that for each $n \geq 1$,

$$\bar{p} \neq \sigma^n.$$

Then $u \neq 0$ by Lemma 4.11.1. The equality (66) then follows similar to the proof of Lemma 4.67. □

**Theorem 4.74.** *The following are equivalent:*

*(1) $R$ is normal.*
*(2) $\sigma S \subset R$.*
*(3) $R = k + \mathfrak{m}_0 S$.*
*(4) $R = k + J$ for some ideal $J$ in $S$.*

*Proof.* We have

- (1) $\Leftrightarrow$ (2) holds by Proposition 4.70.
- (2) $\Leftrightarrow$ (3) holds by Proposition 4.73.
- (3) $\Leftrightarrow$ (4) holds again by Proposition 4.73.

$\square$

**Theorem 4.75.** *The integral closures of the reduced and homotopy centers are isomorphic,*

$$\hat{Z}^c \cong R^c.$$

*Proof.* For brevity, we identify $\hat{Z}$ with its isomorphic $\bar{\psi}$-image in $R$ (Theorem 4.27), and thus write $\hat{Z} \subseteq R$. This inclusion implies

$$\hat{Z}^c \subseteq R^c. \tag{67}$$

To show the reverse inclusion, recall that by Theorem 4.66.3,

$$\operatorname{Frac} \hat{Z} = \operatorname{Frac} R = \operatorname{Frac} S.$$

(i) First suppose $r \in R$. Then there is some $n \geq 1$ such that $r^n \in \hat{Z}$ by Proposition 4.30.3. Whence $r$ is a root of the monic polynomial

$$x^n - r^n \in \hat{Z}[x].$$

Thus $r$ is in $\hat{Z}^c$. Therefore

$$R \subseteq \hat{Z}^c. \tag{68}$$

(ii) Now suppose $s \in R^c \setminus R$. Since $R^c$ is generated by certain monomials in $S$, it suffices to suppose $s$ is a monomial. Then by Claim (2) in the proof of Theorem 4.70, there is a monomial $t \in S$ such that $s = t\sigma$. Furthermore, by Proposition 4.30.2, there is some $N \geq 0$ such that for each $m \geq 1$,

$$t^m \sigma^N \in \hat{Z}.$$

In particular,

$$s^N = t^N \sigma^N \in \hat{Z}.$$

Whence $s$ is a root of the monic polynomial

$$x^N - s^N \in \hat{Z}[x].$$

Thus $s$ is in $\hat{Z}^c$. Therefore, together with (68), we obtain

$$R^c \subseteq \hat{Z}^c. \tag{69}$$

The theorem then follows from (67) and (69). $\square$

**Proposition 4.76.** *The integral closures $R^c \cong \hat{Z}^c$ are nonnoetherian and properly contained in the cycle algebra $S$.*

*Proof.* Since $A$ is non-cancellative, there is some $s \in S$ such that $\sigma \nmid s$ and $s^n \notin R$ for each $n \geq 1$, by Lemma 4.44. In particular, $\sigma \nmid s^n$ for each $n \geq 1$ since $\sigma = \prod_{D \in \mathcal{S}} x_D$. Thus $s$ is not the root of a monic binomial in $R[x]$ by Lemma 4.67. Therefore $s \notin R^c$ since $R$ is generated by monomials in the polynomial ring $B$.

Similarly, $s^m \notin R^c$ for each $m \geq 1$. It follows that $R^c$ is nonnoetherian by Claim (i) in the proof of Theorem 4.45. □

## Appendix A. A brief account of Higgsing with quivers

### *Quiver gauge theories*

According to string theory, our universe is 10 dimensional.[11,12] In many string theories our universe has a product structure $M \times Y$, where $M$ is our usual 4-dimensional space-time and $Y$ is a 6-dimensional compact Calabi-Yau variety.

Let us consider a special class of gauge theories called 'quiver gauge theories', which can often be realized in string theory.[13] The input for such a theory is a quiver $Q$, a superpotential $W$, a dimension vector $d \in \mathbb{N}^{Q_0}$, and a stability parameter $\theta \in \mathbb{R}^{Q_0}$.

Let $I$ be the ideal in $\mathbb{C}Q$ generated by the partial derivatives of $W$ with respect to the arrows in $Q$. These relations (called 'F-term relations') are classical equations of motion from a supersymmetric Lagrangian with superpotential $W$.[14] Denote by $A$ the quiver algebra $\mathbb{C}Q/I$.

According to these theories, the space $X$ of $\theta$-stable representation isoclasses of dimension $d$ is an affine chart on the compact Calabi-Yau variety $Y$. The 'gauge group' of the theory is the isomorphism group (i.e., change of basis) for representations of $A$.

Physicists view the elements of $A$ as fields on $X$. More precisely, $A$ may be viewed as a noncommutative ring of functions on $X$, where the evaluation of a function $f \in A$ at a point $p \in X$ (i.e., representation $p$) is the matrix $f(p) := p(f)$ (up to isomorphism).

### *Vacuum expectation values*

Given a path $f \in A$ and a representation $p \in X$, denote by $f(\bar{p})$ the matrix representing $f$ in the vector space diagram on $Q$ associated to $p$.

A field $f \in A$ is 'gauge-invariant' if $f(p) = f(p')$ whenever $p$ and $p'$ are isomorphic representations (i.e., they differ by a 'gauge transformation'). If $f$ is a path, then $f$ will necessarily be a cycle in $Q$.

The 'vacuum expectation value' of a field is its expected (average) energy in the vacuum (similar to rest mass), and is abbreviated 'vev'. In our case, the vev of a

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[11]Thanks to physicists Francesco Benini, Mike Douglas, Peng Gao, Mauricio Romo, and James Sparks for discussions on the physics of non-cancellative dimers.

[12]More correctly, weakly coupled superstring theory requires 10 dimensions.

[13]Here we are considering theories with $\mathcal{N} = 1$ supersymmetry.

[14]More correctly, the F-term relations plus the D-term relations imply the equations of motion.

path $f \in A$ at a point $p \in X$ is the matrix $f(\bar{p})$, which is just the expected energy of $f$ in $M \times \{p\}$.

*Higgsing*

Spontaneous symmetry breaking is a process where the symmetry of a physical system decreases, and a new property (typically mass) emerges.

For example, suppose a magnet is heated to a high temperature. Then all of its molecules, which are each themselves tiny magnets, jostle and wiggle about randomly. In this heated state the material has rotational symmetry and no net magnet field. However, as the material cools, one molecule happens to settle down first. As the neighboring molecules settle down, they align themselves with the first molecule, until all the molecules settle down in alignment with the first.[15] The orientation of the first settled molecule then determines the direction of magnetization for the whole material, and the material no longer has rotational symmetry. One says that the rotational symmetry of the heated magnet was spontaneously broken as it cooled, and a global magnetic field emerged.[16]

Higgsing is a way of using spontaneous symmetry breaking to turn a quantum field theory with a massless field and more symmetry into a theory with a massive field and less symmetry. Here mass (vev's) takes the place of magnetization, gauge symmetry (or the rank of the gauge group) takes the place of rotational symmetry, and energy scale (RG flow) takes the place of temperature.

The recent discovery of the Higgs boson at the Large Hadron Collider is another example of Higgsing.[17]

*Higgsing in quiver gauge theories*

We now give our main example. Suppose an arrow $a$ in a quiver gauge theory with dimension $1^{Q_0}$ is contracted to a vertex $e$. We make two observations:

(1) the rank of the gauge group drops by one since the head and tail of $a$ become identified as the single vertex $e$;
(2) $a$ has zero vev at any representation where $a$ is represented by zero, while $e$ can never have zero vev since it is a vertex, and $X$ only consists of representation isoclasses with dimension $1^{Q_0}$.

We therefore see that contracting an arrow to a vertex is a form of Higgsing in quiver gauge theories with dimension $1^{Q_0}$.[18]

[15]More precisely, there are domains of magnetization.
[16]This is an example of 'global' symmetry breaking, meaning the symmetry is physically observable.
[17]This is an example 'gauge' symmetry breaking, meaning the symmetry is not an actual observable symmetry of a physical system, but only an artifact of the math used to describe it (like a choice of basis for the matrix of a linear transformation).
[18]This is another example of gauge symmetry breaking.

In the context of a 4-dimensional $\mathcal{N} = 1$ quiver gauge theory with quiver $Q$, the Higgsing we consider in this paper is related to RG flow. We start with a non-superconformal (strongly coupled) quiver theory $Q$ which admits a low energy effective description, give nonzero vev's to a set of bifundamental fields $Q_1^*$, and obtain a new theory $Q'$ that lies at a superconformal fixed point.

*The mesonic chiral ring and the cycle algebra*

The cycle algebra $S$ we introduce in this paper is similar to the mesonic chiral ring in the corresponding quiver gauge theory. In such a theory, the mesonic operators, which are the gauge invariant operators, are generated by the cycles in the quiver. If the gauge group is abelian, then the dimension vector is $1^{Q_0}$. In the case of a dimer theory with abelian gauge group, two disjoint cycles may share the same $\bar{\tau}\psi$-image, but take different values on a point of the vacuum moduli space. These cycles would then be distinct elements in the mesonic chiral ring, although they would be identified in the cycle algebra $S$; see [B3, Remark 3.17].

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